to the memory of Andreas Floer

A SYMPLECTIC FIXED POINT THEOREM FOR TORIC MANIFOLDS

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In this paper, by a toric manifold we mean a nonsingular symplectic quotient $M = \mathbb{C}^n//T^k$ of the standard symplectic space by a linear torus action. Such a toric manifold is in fact a complex Kahler manifold of dimension $n-k$. We denote $p(M)$ and $c(M)$ the cohomology class of the Kahler symplectic form and the first Chern class of M respectively. They both are effective, that is, Poincaré-dual to some holomorphic hypersurfaces. We call a *homology* class in $H_2(M,\mathbb{Z})$ *effective* if it has non-negative intersection indices with fundamental cycles of all compact holomorphic hypersurfaces in M, and denote $\mathcal{E}$ the set of all non-zero effective homology classes. Our main result is the following

Theorem. *Let M be a compact toric manifold with an integer class $p(M)$ of the symplectic form. Then for any hamiltonian diffeomorphism $h: M \to M$*

(i) the number of its fixed points is not less than

$$\max_{\gamma\in\mathcal{E}} \langle c(M),\gamma\rangle / \langle p(M),\gamma\rangle \ ,$$

(ii) the total multiplicity of its fixed points is not less than $\dim H^*(M,\mathbb{C})$.

This formulation deserves some discussion.

1. By definition a hamiltonian diffeomorphism is the time-1 map of a non-autonomous hamiltonian system on M. In his original formulation of the symplectic fixed point conjecture, V.

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Arnold [Ar1],[Ar2] called such h "homologous to the identity". The conjecture itself says that "on a compact symplectic manifold M the number of fixed points of such h is not less than the number of critical points of some function on M." One may consider the theorem as cup-length and Betti-sum estimates for the numbers of fixed points and thus as a confirmation of Arnold's conjecture for toric manifolds. In fact all compact toric manifolds are simply-connected and therefore hamiltonian transformations form just the identity component in the symplectomorphism group of M.

2. The second statement is included in the theorem only for the sake of completeness: It follows from Lefschetz's theorem and holds for any diffeomorphism from the identity component (not necessarily hamiltonian and without the assumption that p is integer of course): any toric manifold has holomorphic cell decompositions and thus its Betti sum equals its Euler characteristic.

3. The first statement of the theorem is absolutely nontrivial. But it is not very general too: it happens very often that $\langle c(M), \gamma\rangle \leq \langle p(M), \gamma\rangle$ for all $\gamma \in \mathcal{E}$, and the conclusion that h *has a fixed point* still follows from Lefschetz's theorem.

Example. Our theorem implies that *a hamiltonian transformation of $S^2 \times S^2$ has at least two fixed points provided that the symplectic areas of the factors have integer ratio* and gives nothing new against Lefschetz's theorem if the ratio is fractional.

4. The most general results so far on symplectic fixed points cover a broad variety of symplectic manifolds satisfying however the following monotonicity restriction: a symplectic manifold M is called *monotone* if its symplectic class and first Chern class are positively proportional on spherical 2-cycles in M:

$$\exists \mu \in [0, \infty] \, \forall \gamma : S^2 \to M \qquad \langle p(M), \gamma\rangle = \mu \langle c(M), \gamma\rangle.$$

Example. $S^2 \times S^2$ is monotone iff the symplectic areas of the factors are equal.

Arnold's conjecture was confirmed for monotone compact symplectic manifolds by

A. Floer [Fl1] for $\mu = 0$ (1987),

A. Floer [Fl2] for $0 < \mu < \infty$ (1989), and

H. Hofer and D. Salamon [HoS] for $\mu = \infty$ (that is, $c(M) = 0$) (1991).

As far as I know our main theorem (along with some results in [HoS]) gives the first examples of non-trivial symplectic fixed point results for *non-monotone* symplectic manifolds.

5. Although all the symplectic fixed point theorems give rise to the Morse-type estimate (for the total multiplicity # of fixed points) that was expected according to Arnold's conjecture ($\# \geq$ Betti-sum(M)), the cup-length-type estimates for the number of geometrically distinct fixed points sometimes lead to weaker results than expected. The first discrepancy of this sort arose in Y.-G. Oh's paper [Oh] on fixed points on $T^{2k} \times \mathbb{C}P^m$ where a lower bound for the fixed point number was found to be $\max(m+1, 2k+1)$ instead of the cup-length bound $m + 2k + 1$ for the critical point number on this manifold. Floer's theorem [Fl2] on strictly monotone symplectic manifolds ($0 < \mu < \infty$) gives another example of this kind. In his theorem the lower estimate for the fixed point number is the greatest common divisor of all values of the first Chern class of the symplectic manifold. Applied to monotone toric manifolds ($c(M) = \mu p(M)$, p is primitive, $\mu \in \mathbb{N}$) this gives $\#(h) \geq \mu$ (which is in fact worse than the cup-length estimate $\dim(M) + 1$ for critical points of functions unless $M \simeq \mathbb{C}P^m$ in which case $\mu = 1$).

We see that our main theorem gives a straightforward generalization of Floer's theorem to non-monotone toric manifolds — and inherits the aforementioned discrepancy as well.

Now let me say a few general words about the proof of our theorem. It is not a secret anymore that symplectic fixed point theorems are Morse-theoretic results for action functionals on spaces of loops in symplectic manifolds. The main difficulty is not that the loop spaces are infinite-dimensional but rather comes from the fact that both Morse index and coindex of action functionals at critical points are infinite. This means that the Morse complex one should construct from the critical points has nothing to do with usual homotopy invariants of the loop spaces. In order to handle the problem one has to construct a sort of *semi-infinite homology theory* (the term comes from graded Lie algebra theory [F], see also [V]).

A general approach to such a homology theory for symplectic loop spaces leads to Floer homology. Floer's construction depends on Gromov's compactness theorem for moduli spaces of holomorphic curves in almost-complex manifolds [Gro] and seems to fail beyond the monotonicity assumption. It is rather a matter of faith whether this failure is only technical.

The idea of the present paper was to try a more elementary approach in a particular case (beyond the monotonicity restriction) where the approach would do. The method we mean is the *finite-dimensional approximation* of action functionals, and it usually works if the symplectic manifold in question can be obtained somehow from a symplectic vector space. For instance, the pioneer Conley-Zehnder theorem [CZ] on symplectic tori (as a twist of fate, they are not toric manifolds) exploits truncations of Fourier series of loops lifted to the universal covering of the torus. Another idea, by B. Fortune and A. Weinstein [FW], was to represent $\mathbb{C}P^m$ as a symplectic reduction of $\mathbb{C}^{m+1}$ by a circle action and to look for critical loops of a suitable invariant action functional in $\mathbb{C}^{m+1}$. We borrow this idea replacing the circle by a torus and thus come to the category of toric manifolds.

As a finite-dimensional approximation we use discrete loops (rather than Fourier truncation) — a method suggested by M. Chaperon [Cha] and Yu.Chekanov [Che] (see also [LS]) — but in a modified form successfully exploited in my preceding papers [Gi1],[Gi2]. However it turns out that in nonmonotone case action functionals do not have a *representative* (in some sense) finite-dimensional approximation, and we face the necessity of constructing the *semi-infinite cohomology as a direct limit over an exhausting sequence of finite-dimensional approximations.*

Another complicating circumstance is that our action functionals are circle-valued maps rather than usual functions, and therefore we deal with *Morse-Novikov theory of multi-valued* *functionals*: our semi-infinite cohomology bears some important algebraic super-structure — an action of a lattice of *covering transformations*.

Furthermore, since our hamiltonian transformations "live" in $\mathbb{C}^n$ instead of $\mathbb{C}^n//T^k$ their fixed points form whole T^k-orbits, and thus we deal with *equivariant Morse theory* and T^k-equivariant cohomology respectively.

Finally, in order to obtain a lower estimate for the number of critical points one needs not only to construct a suitable cohomology but also to prove that it is not trivial or, in other words, to calculate the cohomology (for some special cases at least). It turns out that the toric manifolds are constructive enough in order to make such a computation possible, but the answer and the computation itself depend on all the geometrical combinatorics of Newton polyhedra that usually accompanies the theory of toric manifolds.

One could pronounce the final sentence to our elementary approach in either of the following opposite ways:

- It is fascinatingly profound: it fits together symplectic semi-infinite equivariant Morse-Novikov critical point theory and algebraic geometry on spectra of some cohomological algebras in terms of combinatorial geometry on Newton polyhedra;
- It is discouragingly complicated: it mixes together symplectic, semi-infinite, equivariant, Morse-Novikov critical point theory with algebraic geometry on spectra...

Leaving this choice to reader's taste, I would like to express my gratitude to all participants of the symplectic topology section at the AMS meetings in Baltimore and of the Berkeley-Davis-Santa Cruz-Stanford symplectic seminar where preliminary versions of this work were presented, and especially to Ya. Eliashberg, D. Fuchs, V. L. Ginzburg, D. McDuff, Y.-G. Oh, and A. Weinstein for numerous stimulating discussions.

§1. Toric Manifolds

We recall here a definition and some facts on symplectic toric manifolds. The basic reference for us will be the last chapter of the remarkable book [Au] by M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*. Our approach here to toric manifolds is only a little different from that in the book (basically we dualize notations) and readers will easily recognize corresponding formulations in the book.

Let $\mathbb{C}^n$ denote the standard coordinate 2n-dimensional real vector space provided with the standard symplectic form

$$\text{imaginary part}(-\tfrac{1}{2}\sum dz_i \wedge d\bar{z}_i) \ .$$

The maximal real torus T^n acts on $\mathbb{C}^n$ by hamiltonian linear transformations

$$z_i \mapsto \exp(2\pi\sqrt{-1}t_i)z_i \ , \qquad i = 1, \ldots, n.$$

This action is generated by n quadratic hamiltonians

$$\pi|z_1|^2, \ldots, \pi|z_n|^2 \ .$$

By definition,[1] *a toric symplectic manifold is a symplectic quotient of* $\mathbb{C}^n$ *by a subtorus* $T^k \subset T^n$.

In more detail, one fixes a subtorus $T^k \subset T^n$ and considers the momentum map of its action on $\mathbb{C}^n$,

$$P : \mathbb{C}^n \to \mathbb{R}^{k*} = (\text{Lie } T^k)^* \ .$$

Then one picks a regular value p of the momentum map and defines the corresponding toric orbifold

$$M_p = P^{-1}(p)/T^k \ ,$$

which automatically inherits a symplectic form from $\mathbb{C}^n$.

In even more detail, the momentum map splits as:

[1] As I learned from V. L. Ginzburg (with reference to [KT]) the usual algebraic construction of toric manifolds that begins with a fan in $\mathbb{R}^{n-k}$ is wider than our definition which works only for symplectic toric manifolds.

$$
\begin{array}{ccc}
\mathbb{C}^n & \longrightarrow & \mathbb{R}^{n*} = (\text{Lie } T^n)^* \\
 & & \downarrow \pi \\
 & & \mathbb{R}^{k*} = (\text{Lie } T^k)^*
\end{array}
$$

where π is the projection dual to the embedding Lie $T^k \subset$ Lie T^n, and the horizontal arrow is the momentum map $(\pi|z_1|^2, \ldots, \pi|z_n|^2)$ of the T^n-action. Its image is the first orthant Π in $\mathbb{R}^{n*}$,

$$\Pi = \{t \in \mathbb{R}^{n*} | t_i \geq 0, i = 1, \ldots, n\} \ .$$

Unless the opposite is specified, $\mathbb{R}^n$ and $\mathbb{R}^k$ will always mean Lie T^n and Lie T^k respectively, $\mathbb{Z}^n \subset \mathbb{R}^n$, $\mathbb{Z}^k \subset \mathbb{R}^k$ denote the kernels of the exponential maps $t \mapsto \exp(2\pi t)$: Lie $T^m \to T^m$. Thus $\mathbb{R}^n$ and $\mathbb{Z}^n$ really have the canonical positive basis $e_1, \ldots, e_n$ (while the notations $\mathbb{R}^k$ and $\mathbb{Z}^k$ are a bit frivolous).

Most properties of toric manifolds can be formulated in combinatorial terms of the projection $\pi : \Pi \to \pi(\Pi)$.

1. Regular momentum values.[2] The irregular locus of P consists of projections of all $(k-1)$-dimensional faces of Π.

2. Dimension. For a regular p, $\dim_{\mathbb{R}} P^{-1}(p)/T^k = 2(n-k)$.

3. Compactness. Toric varieties $P^{-1}(p)/T^k$ are compact (they are or are not for all $p \in \pi(\Pi)$ simultaneously) iff

$$(\ker\ \pi) \cap \Pi = \{0\} \ .$$

4. Smoothness. For a regular $p \in \pi(\Pi)$ the toric variety $P^{-1}(p)/T^k$ is non-singular iff projections to $\mathbb{R}^{k*}$ of all the k-dimensional faces of Π that cover p are isomorphisms over $\mathbb{Z}$

[2] Be careful: the corresponding formulation on p.163 in [Au] contains a misprint which can make the following two pages in the book confusing!

(that is the corresponding integer $k \times k$-determinants equal ± 1).

5. Betti sum. The Betti sum of a toric manifold M_p equals the number of the k-dimensional faces of Π that cover p when projected to $\mathbb{R}^{k*}$. In fact it is the number of vertices of the convex polyhedron

$$\pi^{-1}(p) \cap \Pi \subset \mathbb{R}^{n*} \tag{1.1}$$

This polyhedron is identified with the image of the momentum map of the quotient hamiltonian action of T^n/T^k on the toric manifold M_p. Its vertices are in one-to-one correspondence with critical points of some *perfect* Morse function on M_p — the hamiltonian of any dense 1-parametric subgroup in the quotient torus [At],[G].

6. Cohomology. Let $\mathbb{Z}[u]$ denote the *graded* algebra of polynomials on $\mathbb{R}^n$ (with integer coefficients), $u = (u_1, \ldots, u_n)$ be its n generators of *degree* 2. We introduce two ideals in $\mathbb{Z}[u]$:

$I =$ the ideal of $\mathbb{R}^k \subset \mathbb{R}^n$,

$J =$ the ideal of the following union Σ of coordinate $(n-k)$-subspaces in $\mathbb{R}^n$: *a coordinate subspace $\mathbb{R}^{n-k}$ is in Σ if its annihilator $(\mathbb{R}^{n-k})^\perp \subset \mathbb{R}^{n*}$ covers P when projected to $\mathbb{R}^{k*}$.*

Another description:

I is generated by $n-k$ linear equations of $\mathbb{R}^k$ in $\mathbb{R}^n$,

J is spanned by the monomials $u^m = u_1^{m_1} \cdot \ldots \cdot u_n^{m_n}$ whose exponents $m \in \mathbb{Z}^n_+ \subset \mathbb{Z}^n$ being considered as linear functionals on $\mathbb{R}^{n*}$ assume strictly positive values on the (vertices of the) convex polyhedron (1.1).

In fact, J depends only on the component of p in the regular value locus of the momentum map P.

The cohomology ring of the toric manifold M_p is canonically isomorphic to the graded quotient algebra

$$H^*(M_p, \mathbb{Z}) \cong \mathbb{Z}[u]/(I+J) \tag{1.2}$$

The fact that the (complexified) quotient algebra $\mathbb{C}[u]/(I+J)$ has a finite $\mathbb{C}$-dimension reflects the geometric transversality of $\mathbb{R}^k \subset \mathbb{R}^n$ to all the coordinate $(n-k)$-subspaces that constitute Σ (and whose total number equals the Betti sum of M_p).

The isomorphism (1.2) is induced by the composition

$$H^*(BT^n, \mathbb{Z}) \to H^*(BT^k, \mathbb{Z}) \to H^*(M_p, \mathbb{Z})$$

where BT^n, BT^k are classifying spaces of the tori, the left arrow is induced by the embedding $T^k \subset T^n$, the right arrow is induced by the classifying map $M_p \to BT^k$ of the principal T^k-bundle $P^{-1}(p) \to P^{-1}(p)/T^k$, and the characteristic class algebras of T^n and T^k are identified with $\mathbb{Z}[u]$ and $\mathbb{Z}[u]/I$ respectively.

7. Symplectic periods. In particular the above identifications determine epimorphisms

$$\begin{array}{ccc} \mathbb{R}^{k*} & \longrightarrow & H^2(M_p,\ \mathbb{R}) \\ \cup & & \cup \\ \mathbb{Z}^{k*} & \longrightarrow & H^2(M_p,\ \mathbb{Z}) \end{array} \tag{1.3}$$

Using (1.3) we indicate the cohomology class of the symplectic form in the cohomology of the toric symplectic manifold (M_p, ω):

$$[\omega] = \text{ image of } p \in \mathbb{R}^{k*} \text{ under (1.3).}$$

8. Complex structure. More traditional viewpoint on toric varieties is that they are complex-algebraic compactifications of the complex torus $(\mathbb{C}^\times)^{n-k}$. This torus is the quotient $(\mathbb{C}^\times)^n/(\mathbb{C}^\times)^k$ of the complement to the coordinate cross in $\mathbb{C}^n$ by the complexified torus T^k. In order to construct the compactification one defines M_p as a quotient $(\mathbb{C}^n \backslash \mathcal{U}_p)/(\mathbb{C}^\times)^k$ of a bigger complex subspace where

$\mathcal{U}_p =$ union of those coordinate subspaces in $\mathbb{C}^n$

whose momentum image in $\mathbb{R}^{k*}$ does not contain p.

This description provides M_p with a complex structure and stratification which do not depend on p unless p crosses the singular value locus of the momentum map P (when it does the manifold M_p changes itself).

9. First Chern Class. Denote $(e_1^*, \ldots, e_n^*)$ the basis in $\mathbb{R}^{n*}$ dual to the standard basis $(e_1, \ldots, e_n)$ in $\mathbb{R}^n$. Then

$$\mathbb{R}^{k*} \ni \Sigma\pi(e_i^*) \mapsto c(M_p) \in H^2(M_p, \mathbb{Z})$$

where $c(M_p)$ is the first Chern class of the tangent bundle of M_p, and the arrow is the above epimorphism (1.3).

Proof. Generators of the $(\mathbb{C}^\times)^{n-k}$-action on M_p become linear dependent holomorphic vector fields on strata of codimension 1. These strata correspond to hyperplane walls of Π and form a divisor in M_p Poincaré-dual to $c(M)$.

Notice that a codimension-1 stratum is empty iff the corresponding wall $\langle e_i \rangle^\perp$ does not cover p in which case however $\pi(e_i^*)$ is in the kernel of the epimorphism (1.3) too.

The fact that a wall $\langle e_i \rangle^\perp$ does not cover p actually means that the representation of M_p as a toric manifold is not *minimal* in the following sense: one can obtain the same symplectic manifold as a symplectic quotient of $\mathbb{C}^{n-1}$ by T^{k-1}. Thus without loss of generality *we will assume further that our toric manifold M_p is minimal, that is, p is the image of all hyperplane walls of* Π, *and therefore (1.3) is an isomorphism*

$$\mathbb{R}^{k*} \simeq H^2(M_p, \mathbb{R}) \ .$$

10. Effective classes. The set $\mathcal{E} \subset \mathbb{Z}^k = H_2(M_p, \mathbb{Z})$ of effective homology classes is the intersection of the lattice with the 1-st orthant in $\mathbb{R}^n$:

$$\gamma \in \mathcal{E} \Longleftrightarrow \langle e_i^*, \gamma \rangle \geq 0 \quad \text{for } i = 1, \ldots, n. \tag{1.4}$$

Indeed, any holomorphic hypersurface in M_p contracts to a positive linear combination of the codimension-1 strata by the action of the imaginary part of the complex quotient torus $T^n_{\mathbb{C}}/T^k_{\mathbb{C}}$, and these strata are Poincaré dual to $\pi(e^*_i)$.

Example. $S^2 \times S^2$. Let an action of T^2 on $\mathbb{C}^4$ be generated by $\pi(|z_1|^2 + |z_2|^2)$ and $\pi(|z_3|^2 + |z_4|^2)$. The corresponding projection $\pi : \mathbb{R}^{4*} \to \mathbb{R}^{2*}$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The image of the first orthant and the corresponding first Chern class are shown in Figure 1. The toric manifold $P^{-1}(p)/T^2$ is $S^2 \times S^2$ provided with a product symplectic form $\omega = \omega_1 \oplus \omega_2$ with periods

$$p = (\int_{S^2} \omega_1, \int_{S^2} \omega_2).$$

It is monotone only if the areas of factors are equal. Effective homology classes form the first quadrant on the dual lattice $\mathbb{Z}^2$.

It is convenient sometimes to choose a basis in $\mathbb{Z}^k$ and determine the projection $\pi : \mathbb{R}^{n*} \to \mathbb{R}^{k*}$ by an integer $k \times n$-matrix $\pi = (\pi_{ij})$ as in the example. In terms of this matrix, compactness of the toric manifold means that upon a suitable choice of the basis in $\mathbb{Z}^k$ the matrix has non-negative entries with positive sum in each column. The 1-st Chern class $c(M_p)$

is represented in $\mathbb{R}^{k*}$ by the total sum of all columns. Smoothness of M_p (for regular p) means that each k-minor of the matrix such that p is the convex hull of its columns is unimodular ($\det = \pm 1$).

Now we are ready to give a combinatorial reformulation of our main theorem.

Theorem 1.1. *Let $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n \cap \mathbb{Z}^k$ be a non-negative point in the lattice $\mathbb{Z}^k \subset \mathbb{Z}^n$, $m_i \geq 0$, $m \neq 0$, $p \in \mathbb{Z}^{k*}$ be a primitive (that is, $GCD(\langle p, \ell\rangle | \ell \in \mathbb{Z}^k) = 1$) regular value of $\pi : \Pi \to \mathbb{R}^{k*}$. Then for any hamiltonian diffeomorphism $M_p \to M_p$ the number of its fixed points is not less than $(m_1 + \cdots + m_n)/\langle p, m\rangle$.*

Example. $S^2 \times S^2$. For $p = (p_1, p_2)$ with $p_1 \geq p_2 > 0$, $m = (m_1, m_1, m_2, m_2)$ Theorem 1 gives rise to

$$2(m_1 + m_2)/(p_1 m_1 + p_2 m_2) = 2/\left[p_2 + \frac{(p_1 - p_2)m_1}{m_1 + m_2}\right] \leq 2$$

where the equality holds only if $p_2 = 1$ and $m_1 = 0$ (or $p_1 = p_2$).

Our proof of Theorem 1.1 in §6 is based on the machinery of semi-infinite equivariant Morse theory developed in §§$2 - 6$ and on properties of some cohomological algebras described below.

Let $\mathbb{C}[u, u^{-1}] = \mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$ be the algebra of polynomial functions on the complex torus $\mathbb{C}^n \backslash$(coordinate cross) $= (\mathbb{C}\backslash 0)^n$. It contains the polynomial algebra $\mathbb{C}[u]$ and we will treat $\mathbb{C}[u, u^{-1}]$ as a $\mathbb{C}[u]$-module. Let us introduce the following submodules in $\mathbb{C}[u, u^{-1}]$:

I = the ideal of $\mathbb{C}^k \subset \mathbb{C}^n$ in $\mathbb{C}[u]$ (here $\mathbb{C}^k \subset \mathbb{C}^n$ is the complexification of $\mathbb{R}^k \subset \mathbb{R}^n$);

$J_r = (u^m, m \in \mathbb{Z}^k | \langle p, m\rangle \geq r)_{\mathbb{C}[u]}$ (here p is a regular integer primitive value of $\pi : \Pi \to \mathbb{R}^{k*}$).

Proposition 1.2. *The quotient algebra* $\mathbb{C}[u]/(I+\mathbb{C}[u]\cap J_r)$ *is finite dimensional.*

Proof. It is a funny combination of Hilbert's zeroes theorem with geometry on Newton lattices. Let us think of integer points $m\in\mathbb{Z}^n$ in $\mathbb{C}^n$ as of exponents of basic monomials $u^m\in\mathbb{C}[u,u^{-1}]$. This "Newton lattice" $\mathbb{Z}^n$ contains a sublattice $\mathbb{Z}^k$ (integer points in $\mathbb{C}^k\subset\mathbb{C}^n$) and a half-lattice

$$\mathbb{Z}^k_r=\{m\in\mathbb{Z}^k|\langle p,m\rangle\geq r\}$$

of monomials generating J_r. We assert at first that *if* $\mathbb{C}^k$ *has a nontrivial intersection with a coordinate subspace* $\mathbb{C}^d\subset\mathbb{C}^n$ *then the halflattice* $\mathbb{Z}^k_r$ *does too.* Indeed, if $\mathbb{Z}^k_r\cap\mathbb{C}^d=\varnothing$ then $p|\mathbb{C}^k\cap\mathbb{C}^d=0$, and if $\dim\mathbb{C}^k\cap\mathbb{C}^d=\ell>0$ then the π-image in $\mathbb{R}^{k*}$ of the orthogonal coordinate subspace $(\mathbb{R}^d)^\perp\subset\mathbb{R}^{n*}$ has positive codimension ℓ and contains p, in which case p would be an *irregular* value of $\pi:\Pi\to\mathbb{R}^{k*}$.

Now let us consider a coordinate subspace $\mathbb{C}^d$ that does meet $\mathbb{Z}^k_r$ at some point m_0 (see Figure 2). The first orthant in $\mathbb{Z}^d\subset\mathbb{C}^d$ intersects with the first orthant shifted to m_0. This means that $(u^{m_0}\mathbb{C}[u])\cap\mathbb{C}[u]$ *contains a monomial that* (as a function on $\mathbb{C}^d$) *does not vanish in* $\mathbb{C}^d\backslash$(coordinate cross).

Now let us consider in $\mathbb{C}^n$ the zero set of the ideal $\mathbb{C}[u] \cap J_r$. What has been said implies that this set is contained in the union of those coordinate subspaces $\mathbb{C}^d$ in $\mathbb{C}^n$ which have trivial intersection with $\mathbb{C}^k$. In other words: *the zero set of* $I + \mathbb{C}[u] \cap J_r$ *contains at most one point — the origin.* Now Hilbert's theorem completes the proof.

Later we will apply this information in the following form. Let $\mathcal{J}_r$ mean the image of J_r in the quotient algebra $\mathcal{R} = \mathbb{C}[u, u^{-1}]/\mathbb{C}[u, u^{-1}]I$, $\mathcal{J}$ a $\mathbb{C}[u]$-submodule such that $\mathcal{J}_{r_+} \subset \mathcal{J} \subset \mathcal{J}_{r_-}$ for some $r_+ > r_-$.

Corollary 1.3. *There exists $q \in \mathcal{R}$ such that $q \notin \mathcal{J}$ but $u_1 q, \dots, u_n q \in \mathcal{J}$.*

Indeed, obviously $1 \notin J_1$ and thus there is a monomial u^{-m} which is not in $J_{r_-} + I\mathbb{C}[u, u^{-1}]$. Since

$$u^{-m}\mathbb{C}[u]/(J_{r_+} \cap u^{-m}\mathbb{C}[u] + u^{-m}I)$$

is finite-dimensional one can choose q as a monomials of maximal degree in $u^{-m}\mathbb{C}[u]$ among those whose image in $\mathcal{R}$ is still not in $\mathcal{J}$.

Remark 1.4. The total Betti sum estimate of the total multiplicity of fixed points could also be obtained by Morse-theoretic tools. But instead of the cohomology algebra $\mathbb{C}[u]/(I + J)$ of the toric manifold (See (1.2)) one would then face another interesting "semi-infinite cohomology algebra", namely $\mathcal{J}_0/\mathcal{J}_1$. It is a module (probably free) over

$$R_p = \mathrm{Span}(u^m, m \in \mathbb{Z}^k \subset \mathbb{Z}^n | \langle p, m \rangle = 0)$$

and its rank equals the total Betti sum of the toric manifold M_p.

§2. Least Action Principle

The principle says that fixed points of hamiltonian transformations correspond to critical points of action functionals on loop spaces. Following [FW] we intend to consider loops in $\mathbb{C}^n$ lifting a hamiltonian isotopy of a toric manifold $M_p = \mathbb{C}^n//T^k$ up to a homogeneous equivariant hamiltonian isotopy of $\mathbb{C}^n$. In this section we formulate the infinite dimensional Morse - theoretic problem that arises in this way. By *homogeneity* we mean $\mathbb{R}_+^\times$-equivariance with respect to the dilatation group action in $\mathbb{C}^n$, and we will talk of homogeneous functions, vector fields and diffeomorphisms in $\mathbb{C}^n$ keeping in mind that they are smooth probably only in $\mathbb{C}^n \backslash 0$.

Proposition 2.1. *A hamiltonian isotopy h^t of a compact toric manifold $M_p = \mathbb{C}^n//T^k$ can be lifted up to a T^k-equivariant homogeneous hamiltonian isotopy in $\mathbb{C}^n$.*

Proof. Consider the Poisson quotient map $\mathbb{C}^n \to \mathbb{C}^n/T^k$ to the Poisson variety $\mathbb{C}^n/T^k$. Its Cazimir functions are components of the momentum map $P : \mathbb{C}^n \to \mathbb{C}^n/T^k \to \mathbb{R}^{k*}$ and its symplectic leaves are our toric varieties $M_s, \quad s \in \mathbb{R}^{k*}$. We start with a hamiltonian function on a smooth symplectic leaf M_p and extend it somehow to a regular function on $\mathbb{C}^n/T^k$ homogeneous of degree 2 with respect to $\mathbb{R}_+^\times$-action. The extension is possible due to the fact that near M_p the symplectic leaves form a fibration. Then we pull back the extended hamiltonian function to $\mathbb{C}^n$. The lifted function is homogeneous of degree 2 in $\mathbb{C}^n$ and T^k-invariant. This means that its hamiltonian flow commutes with $\mathbb{R}_+^\times$- and T^k-actions, preserves $P^{-1}(p)$ and thus projects to the original flow on $M_p = P^{-1}(p)/T^k$.

Now let $\mathcal{H}^t$ denote a homogeneous, degree 2, T^k-invariant hamiltonian on $\mathbb{C}^n \times [0,1]$. We define *the action functional*

$$\mathcal{A} : \mathcal{L}\mathbb{C}^n \times \mathbb{R}^k \to \mathbb{R} \tag{2.1}$$

$$\mathcal{A} = \oint pdq - \oint \mathcal{H}^t dt - \lambda_1 \oint \mathcal{P}_1 dt - \cdots - \lambda_k \oint \mathcal{P}_k dt$$

on the product of the loop space $\mathcal{L}\mathbb{C}^n$ and $\mathbb{R}^k = \text{Lie } T^k$, where pdq is a potential for the symplectic form in $\mathbb{C}^n$, $\mathcal{P} =$

$(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ are components of the momentum map of the T^k-action on $\mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ are Lagrange multipliers. The action functionals

$$\mathcal{A}_\lambda = \mathcal{A}|\mathcal{L}\mathbb{C}^n \times \{\lambda\} \tag{2.2}$$

are homogeneous degree 2 T^k-invariant functions on the vector space $\mathcal{L}\mathbb{C}^n$. We are going to indicate some relation between critical points of $\{\mathcal{A}_\lambda\}$ and fixed points of the time-1 hamiltonian transformation h induced by the hamiltonians $\mathcal{H}^t$ on the toric manifold M_p.

At first a fixed point corresponds to a time-1 closed trajectory of the hamiltonian isotopy h^t on M_p ($h = h^1$). Since h^t is lifted up to a T^k-invariant isotopy in $\mathbb{C}^n$ its trajectory in M_p is lifted to a trajectory in $P^{-1}(p)$. Such a lifting is unique up to a shift by the T^k-action (so it is not unique!) Even if the trajectory in M_p is closed its liftings can be disclosed. But they become closed if we replace $\mathcal{H}^t$ by $\mathcal{H}^t + \langle \mathcal{P}, \lambda_0 \rangle$ with appropriate λ_0. This modification means that the $\mathcal{H}^t$-isotopy is accompanied by the action of the 1-parametric subgroup $\exp(t\lambda_0) \subset T^k$, generated by $\lambda_0 \in \mathbb{R}^k = \text{Lie } T^k$.

The choice of such a "closing" λ_0 is not unique: the resulting time-1 transformation $\exp(\lambda_0)$ will not change if we add to λ_0 any integer vector $m \in \mathbb{Z}^k \subset \mathbb{R}^k$.

Summarizing, we can say that a fixed point of h on M_p corresponds to a $\mathbb{Z}^k$-lattice of T^k-orbits of closed trajectories in $P^{-1}(p)$ of hamiltonians $\mathcal{H}^t + \langle \mathcal{P}, \lambda_0 + \mathbb{Z}^k \rangle$.

According to the Least Action Principle closed trajectories of hamiltonians $\mathcal{H}^t + \langle \mathcal{P}, \lambda \rangle$ are critical points of action functionals $\mathcal{A}_\lambda$ on $\mathcal{L}\mathbb{C}^n$. Due to homogeneity of $\mathcal{A}_\lambda$ all the critical points have critical value 0 (Euler's formula for homogeneous functions). Now we have to extract those critical loops that are situated in $P^{-1}(p)$.

Let $\mathcal{S}$ denote the sphere of all rays ($\mathbb{R}_+^\times$-orbits) in the loop space $\mathcal{L}\mathbb{C}^n \backslash \{\text{zero loop}\}$, $A_\lambda \subset \mathcal{S}$ — (the rays on) the zero cone of the homogeneous function $\mathcal{A}_\lambda$, $A \subset \mathcal{S} \times \mathbb{R}^k$ is the union of $\{A_\lambda\}_{\lambda \in \mathbb{R}^k}$,

$$A = [\mathcal{A}^{-1}(0) \backslash (0 \times \mathbb{R}^k)]/\mathbb{R}_+^\times \; . \tag{2.3}$$

We pull back the linear function $p : \mathbb{R}^k \to \mathbb{R}$ to $\mathcal{S} \times \mathbb{R}^k$ and denote $\hat{p} : A \to \mathbb{R}$ its restriction to A. Both A and $\hat{p}$ are T^k-invariant.

Proposition 2.2. *Fixed points of $h : M_p \to M_p$ are in one-to-one correspondence with $\mathbb{Z}^k$-lattices of critical T^k-orbits of the function $\hat{p}$ (see Figure 3)*

Proof. Let us notice first that A is nonsingular. Indeed, critical points of $\mathcal{A}$ are those closed trajectories of hamiltonians $\mathcal{H}^t + \langle \mathcal{P}, \lambda \rangle$ which satisfy the constraint $\oint \mathcal{P} = 0$. A critical trajectory is situated on a level of the momentum map (since $\mathcal{H}^t$ are T^k-invariant) and the constraint

$$\oint \mathcal{P} = \overrightarrow{\mathrm{const}} \tag{2.4}$$

implies the point-wise one

$$\mathcal{P} = \overrightarrow{\mathrm{const}} \tag{2.5}$$

In particular if $\overrightarrow{\text{const}} = 0$ the trajectory must be the zero loop since $P^{-1}(0) = 0$ (compactness of our toric manifolds!, see §1, point 3).

Now, a ray in $\mathcal{A}^{-1}(0) \times \mathbb{R}^k$ is critical for $\hat{p}$ if and only if it is critical for corresponding $\mathcal{A}_\lambda$ (recall that non-zero critical points of $\mathcal{A}_\lambda$ always form rays on the zero cone $\mathcal{A}_\lambda = 0$) and the derivative of $\mathcal{A}$ in λ is proportional to p, that is, along this ray

$$\oint \mathcal{P} \sim p \ .$$

Together with (2.5) this means that the ray contains a closed trajectory satisfying $\mathcal{P} \equiv p$, and vice versa.

§3. Fronts and their generating families

We formulate here some technical statements allowing us to carry out homotopies in families of functions looking at their discriminants. This technique containing transversality and Morse-theoretic arguments is clear by itself and becomes especially obvious after the book *Stratified Morse Theory* by M. Goresky and R. McPherson [GM] where we refer the reader for details.

Let us consider a family $f : X \times \Lambda \to \mathbb{R}$ of functions f_λ on a compact manifold X where $\Lambda \ni \lambda$ is a parametrizing manifold. In our applications Λ will be simply a Euclidean space, and $f \in C^1$ with Lipschitz derivatives (so that Morse-theoretic gradient flow deformations still apply).

We will consider restrictions of such a family to compact submanifolds $\Gamma \subset \Lambda$ with boundary $\partial\Gamma$ and study homotopy types of sets

$$F_\Gamma^+ = \{(x, \lambda) \in X \times \Lambda | \lambda \in \Gamma,\ f(x, \lambda) \geq 0\}$$
$$F_\Gamma^- = \{(x, \lambda) \in X \times \Lambda | \lambda \in \Gamma,\ f(x, \lambda) \leq 0\}$$

Assuming that $f^{-1}(0)$ is non-singular let us define *the front* Φ of the family f:

$$\Phi = \{\lambda \in \Lambda | f_\lambda^{-1}(0) \text{ is singular}\}$$

It is better to think of the front as of a singular hypersurface in Λ provided at every point with tangent hyperplane(s) (not unique in general). In fact, the front can be obtained by the following contact geometry construction [AVG], v.1. In the contact manifold $PT^*(X \times \Lambda)$ of all *contact elements* (= tangent hyperplanes) on $X \times \Lambda$ we consider a Legendrian submanifold L of such elements tangent to the hypersurface $f^{-1}(0)$. Intersection of L with the submanifold $\mathcal{P}$ of all *vertical* contact elements (those tangent to fibers of $X \times \Lambda \to \Lambda$) is, generally speaking, a subvariety $\hat{\Phi}$ in $\mathcal{P}$. This subvariety parametrizes the front Φ in the following two-step way. At first we project $\hat{\Phi}$ to the space $PT^*(\Lambda)$ of contact elements on Λ ($\mathcal{P} \to PT^*(\Lambda)$ projects a vertical contact element on $X \times \Lambda$ to a contact element on Λ). The image $\hat{L}$ is a Legendrian subvariety of the contact structure on $PT^*(\Lambda)$ in the sense that it is integral at its non-singular points. Then $\hat{L}$ projects to the front Φ in the base Λ. By a tangent hyperplane to Φ at Λ we mean a contact element from $\hat{L} \cap PT^*_\lambda(\Lambda)$.

For *generic* f its front Φ actually is a hypersurface because $\hat{L}$ is an immersed Legendrian submanifold in this case. In general if f were smooth Φ would have zero measure due to Sard's lemma (Φ is the critical value locus for the projection $f^{-1}(0) \to \Lambda$). Our C^1-assumption is not sufficient for that but in our applications Φ will still have zero measure since all critical points of $f^{-1}(0) \to \Lambda$ will appear to be C^∞-points of $f^{-1}(0)$.

We call a submanifold in Λ *transversal* to the front Φ if at its every intersection point with Φ it is transversal to all the tangent hyperplanes to Φ.

From the definitions we obtain

Proposition 3.1. *A submanifold $\Gamma \subset \Lambda$ is transversal to the front Φ if and only if $X \times \Gamma$ is transversal to $f^{-1}(0)$.*

Corollary 3.2. *Suppose that f is C^∞ at all critical points with zero critical values of all functions f_λ. Let $\Gamma_t = \rho^{-1}(t)$ be non-singular levels of some smooth map $\rho : \Gamma \to \mathbb{R}^m$. Then* *almost all Γ_t are transversal to Φ.*

Applying standard Morse theoretic arguments we come to the following proposition.

Proposition 3.3. *Suppose that f and Γ deform in a way that $f^{-1}(0)$ remains nonsingular and both Γ and $\partial\Gamma$ remain transversal to the front of f. Then homotopy types of F^+_Γ, F^-_Γ do not change during the deformation.*

Corollary 3.4. *If f varies in a way that its front Φ does not change, and Γ and $\partial\Gamma$ are transversal to Φ, then homotopy types of $F^\pm_\Gamma$ do not change.*

Remarks 3.5. (1) If f is invariant under a fiberwise action of a compact Lie group (say a torus) on X then all the mentioned homotopies can be chosen equivariant.

(2) Under assumptions made, not only homotopy types of $F^\pm_\Gamma$ remain unchanged but also those of the subdivision $X \times \Gamma = F^+_\Gamma \cup F^-_\Gamma$ by $f^{-1}(0)$, so that we can replace $F^\pm_\Gamma$ in the proposition and its corollary by the pairs $(F^\pm_\Gamma, F^\pm_{\partial\Gamma})$, or by $f^{-1}_\Gamma(0)$ (or by something else). We will refer to homotopy types of all such spaces as *the homotopy type of the function* $f|_{X\times\Gamma}$ itself, keeping in mind that it is the homotopy type with respect to its *zero level* only.

(3) All the formulated statements remain valid if we replace the manifold Γ with boundary by a more complicated stratified manifold (say the surface of a cube). In thise case one should improve the transversality definition: Γ is said to be transversal to Φ if each of its strata is.

Example: Action functionals. We may consider the family

$$\mathcal{A} = \{\mathcal{A}_\lambda\} : \mathcal{S} \times \mathbb{R}^k \to \mathbb{R}$$

of action functionals restricted to the unit sphere $\mathcal{S}$ in the loop space of $\mathcal{L}\mathbb{C}^n$ (at least formally — $\mathcal{S}$ is infinite-dimensional and non-compact). Its front $\Phi \subset \mathbb{R}^k$ consists of those λ for which the time-1 map of the hamiltonian $\mathcal{H}_t + \langle \mathcal{P}, \lambda\rangle$ has non-trivial fixed points. It will be essential later that the front is periodic in the sense that it is the lifting to $\mathbb{R}^k$ of some front in $\mathbb{R}^k/\mathbb{Z}^k$ (notice that the family $\mathcal{A}_\lambda$ is *not* periodic in λ).

Roughly speaking we will use the fact that the front in $\mathbb{R}^k/\mathbb{Z}^k$ has "bounded geometry" since it is compact.

According to Propositions 2.2 and 3.1 fixed points of our original hamiltonian transformation on M_p correspond to tangency events of levels $p = \text{const}$ of the linear function $p : \mathbb{R}^k \to \mathbb{R}$ with the front Φ. For integer p the tangency points form whole $(k-1)$-dimensional lattices (due to periodicity of Φ) and our objective will be to "count" the number of such tangency lattices between two nonsingular levels, say $p^{-1}(0)$ and $p^{-1}(1)$.

§4. Generating functions of hamiltonian maps

In this section we construct finite-dimensional approximations to action functionals described in §2. Given a hamiltonian isotopy h^t of a symplectic manifold (M, ω) we (following M. Chaperon [Cha]) represent it as a composition $h_N \circ \cdots \circ h_1$ of "small" hamiltonian transformations and (following [Gi1]) define two symplectomorphisms $h^{(N)}, q^{(N)}$ of $M^{(N)} = (M \times \cdots \times M, \omega \oplus \cdots \oplus \omega)$ to itself:

$$h^{(N)} = (h_1, h_2, \dots, h_N) \quad \text{the component-wise map}$$

$$q^{(N)} : (x_1, \dots, x_N) \mapsto (x_2, \dots, x_N, x_1) \quad \text{the cyclic shift.}$$

Obviously fixed points of the time-1 map h^1 are in one-to-one correspondence with solutions of the equation $q^{(N)}(\vec{x}) = h^{(N)}(\vec{x})$ and with intersection points of graphs of $q^{(N)}$ and $h^{(N)}$. These graphs are Lagrangian submanifolds in $(M^{(N)} \times \bar{M}^{(N)}, \omega^{(N)} \ominus \omega^{(N)})$ and the second one is close to the diagonal $\Delta \subset M^{(N)} \times \bar{M}^{(N)}$. A neighborhood of Δ is symplectomorphic to $T^*\Delta$ and thus the graph H of $h^{(N)}$ represents in $T^*\Delta$ the differential of some function $\mathcal{H} : \Delta \to \mathbb{R}$. (This function is called the *generating function* of H or $h^{(N)}$.) Now assume that $(M, \omega) \simeq \mathbb{C}^n$. Then $M^{(N)} \times \bar{M}^{(N)}$ has a global structure of the cotangent bundle $T^*\Delta$ and the graph Q of the cyclic shift also has a generating function $\mathcal{Q}$. Thus the fixed points correspond to critical points of $\mathcal{F} = \mathcal{Q} - \mathcal{H}$. This is the idea of our construction.

In our actual situation described in §2, we begin with a *homogeneous* hamiltonian isotopy $h^t : \mathbb{C}^n \to \mathbb{C}^n$ and subdivide it into the *even* number $2N$ of "small" parts $(h_1, \dots, h_{2N})$ so that $H = H_1 \times \cdots \times H_{2N}$ is a *conical* Lagrangian graph of the componentwise map $(\mathbb{C}^n)^{2N} \to (\mathbb{C}^n)^{2N}$. By a technical reason we define $q^{(N)}$ as the cyclic shift twisted by the central symmetry:

$$q(z_1, \dots, z_{2N}) = (z_2, \dots, z_{2N}, -z_1), \qquad z_1, \dots z_{2N} \in \mathbb{C}^n \ ,$$

so that $Q =$ graph $q^{(N)}$ is a Lagrangian subspace in $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N}$.

Now we introduce in $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N}$ a structure of the cotangent bundle over the diagonal $\Delta (\simeq \mathbb{C}^{2nN})$ choosing Δ itself for its zero section and $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N} \to \Delta$ along the antidiagonal

$$(z_1, w_1, \dots, z_{2N}, w_{2N}) \mapsto \left(\frac{z_1 + w_1}{2}, \dots, \frac{z_{2N} + w_{2N}}{2} \right)$$

for the canonical projection $T^*\Delta \to \Delta$. What was said above implies that $\mathcal{F} = Q - \mathcal{H}$ is a well defined *homogeneous degree 2 function on Δ whose critical points are in one-to-one correspondence with fixed points of the hamiltonian transformation* $(-\mathrm{id}) \circ h^1$. We call it the *generating function* of h^1.

Remarks 4.1. (1) $\mathcal{Q}$ is a non-degenerate quadratic form on Δ: it follows from the fact that the Lagrangian subspace Q is transversal to the diagonal and anti-diagonal since id and $(-\mathrm{id})^{2N+1}$ do not have non-trivial fixed points (this justifies our choice of "even" subdivisions and the "twisting").

(2) $\mathcal{H}$ is well defined provided that $h_1, \dots, h_{2N}$ are sufficiently close to identity and is a homogeneous degree 2 C^1-function with Lipschitz derivatives since $H_1, \dots, H_{2N}$ are conical Lagrangian subspaces in $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ smooth outside the origin. This implies also that $\mathcal{H}$ is C^∞ outside the "coordinate cross" in $(\mathbb{C}^n)^{2n}$.

(3) Notice that a critical point of $\mathcal{F} = \mathcal{Q} - \mathcal{H}$, that is, a discrete loop

$$(z_1, w_1, z_2, w_2, \ldots, z_{2N}, w_{2N}) =$$
$$(z_1, h_1(z_1), h_1(z_1), h_2(h_1(z_1)), \ldots, h_{2N}(\ldots(h_1(z_1)\ldots) = -z_1)$$

coincides with the zero loop if at least one of the coordinates $z_i - w_i$ vanishes (since h_i are small). This means that $\mathcal{F}$ is smooth (C^∞) at every non-trivial critical point.

Proposition 4.2. *Suppose that all the transformations $h_1, \ldots, h_{2N}$ commute with the T^k-action on $\mathbb{C}^n$. Then the generating function $\mathcal{F} : \Delta \simeq (\mathbb{C}^n)^{2N} \to \mathbb{R}$ is T^k-invariant relative to the simultaneous action of the torus on all 2N factors $\mathbb{C}^n$.*

Proof. All the ingredients of the above construction — the symplectic structure in $\mathbb{C}^n$, graphs of h_i, $\pm\mathrm{id}$: $\mathbb{C}^n \to \mathbb{C}^n$ and of the cyclic shift — are invariant with respect to the simultaneous action of the torus on all factors in $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N}$.

In particular this proposition means that critical points of $\mathcal{F}$ actually occur as whole T^k-*orbits of critical rays* (due to the homogeneity and T^k-invariance) similar to those of action functionals $\mathcal{A}_\lambda$.

Notice also that $-\mathrm{id} : \mathbb{C}^n \to \mathbb{C}^n$ commutes with T^k-action and therefore induces symplectic transformations on toric manifolds M_p. Such a transformation is *hamiltonian*. This means that any estimate for fixed points of all its compositions with hamiltonian transformations on M_p will give rise to the same estimate for all hamiltonian transformations themselves: our twisting does not affect the resulting supply of symplectomorphisms.

The generating function $\mathcal{F}$ that we have constructed serves as a finite-dimensional approximation for a single action functional $\mathcal{A}_0$. Now we will take into account the effect of the Lagrange multipliers.

For a given $\lambda \in \mathbb{R}^k = \mathrm{Lie}\, T^k$ we put $t = \exp(\lambda/2N_1)$ where N_1 is big enough to make the transformation t sufficiently close to identity and apply the above construction of the generating function to the decomposition

$$h_{2N_2} \circ h_{2N_2-1} \circ \cdots \circ h_2 \circ h_1 \circ \underbrace{t \circ \cdots \circ t}_{2N_1 \text{ times}}$$

of our time-1 hamiltonian transformation $h^1 : \mathbb{C}^n \to \mathbb{C}^n$ composed with linear transformation $\exp(\lambda)$ (and twisted by $-\operatorname{id}$ of course). We denote the resulting generating function as

$$\mathcal{F}_\lambda^{(N)} : (\mathbb{C}^n)^{2N} \to \mathbb{R}, \qquad N = N_1 + N_2.$$

It will serve us as a finite-dimensional substitute for the action functional $\mathcal{A}_\lambda$.

What we can see from this construction is that it does not work *simultaneously* for all $\lambda \in \mathbb{R}^k$: N_1 must grow with λ. We will assume further that for given h^1 its decomposition $h_{2N_2} \circ \cdots \circ h_1$ is fixed, and for given $N > N_2$ the family $\mathcal{F}_N = \{\mathcal{F}_\lambda^{(N)}\}$ is defined over a compact domain $\Lambda_N \subset \mathbb{R}^k$ of parameter values. For the sake of definiteness let us consider Λ_N to be a cube in $\mathbb{R}^k$ (centered at the origin) of the size growing linearly with N. Thus we come to *an exhausting sequence of generating families*

$$\mathcal{F}_N : \mathbb{C}^{2Nn} \times \Lambda_N \to \mathbb{R}, \qquad \Lambda_N \subset \mathbb{R}^k, \quad \underset{N}{\cup} \Lambda_N = \mathbb{R}^k.$$

It is a family of homogeneous degree 2, T^k-invariant functions on $(\mathbb{C}^n)^{2N}$ provided with the component-wise action of the torus.

Now let $\mathcal{S}_N$ denote a unit sphere S^{4nN-1} in $\mathbb{C}^{2Nn}$ or, to be more invariant, the sphere of all real rays in $\mathbb{C}^{2Nn} \backslash 0$. Each homogeneous function $\mathcal{F}_\lambda^{(N)}$ determines a subdivision of $\mathcal{S}_N$ into positive and negative parts

$$F_N^{\pm} = \{(x, \lambda) \in \mathcal{S}_N \times \Lambda_N | \quad \mathcal{F}_\lambda^{(N)}|_x \geq 0 (\text{resp.} \leq 0)\}$$

and their intersection

$$F_N^0 = (\mathcal{F}_N^{-1}(0)\backslash\Lambda_N)/\mathbb{R}_+^\times = F_N^+ \cap F_N^- \ .$$

We denote $\hat{p}_N$ the following composed function on F_N^0:

$$\hat{p}_N : F_N^0 \subset \mathcal{S}_n \times \Lambda_N \to \Lambda_N \subset \mathbb{R}^k \xrightarrow{p} \mathbb{R}$$

where $p \in \mathbb{R}^{k*} = (\mathrm{Lie}\, T^k)^*$ is the linear function that determines our toric manifold M_p.

Proposition 4.3. *F_N^0 is a non-singular submanifold in $\mathcal{S}_n \times \Lambda_N$. Critical T^k-orbits of the function $\hat{p}_N$ correspond to fixed points of the hamiltonian transformation $-\mathrm{id} \circ h_{2N_2} \circ \cdots \circ h_1$ pushed forward to M_p.*

Remark. The correspondence is not one-to-one. Actually the critical orbits of the functions $\hat{p}_N$ fit some $\mathbb{Z}^k$-lattices (and exhaust them as $N \to \infty$) in the following sense: a fixed point of our hamiltonian transformation on M_p corresponds to a finite set of critical T^k-orbits of $\hat{p}_N$ on F_N^0. This critical set is situated in $\mathcal{S}_N \times \Lambda_N$ over the part of some lattice $\lambda_0 + \mathbb{Z}^k$ in $\mathbb{R}^k$ that fits the cube $\Lambda_N \subset \mathbb{R}^k$. Proposition 4.3 reduces our symplectic fixed point problem to the lower estimate problem for the number of such lattices in $\mathbb{R}^k$. This is a sort of equivariant Lusternik-Schnirelman-Morse-Novikov problem.

Proof. We will look for non-zero critical points of the function

$$\mathcal{F}_N : \mathbb{C}^{2Nn} \times \Lambda_N \to \mathbb{R}$$

and of its restrictions to levels of the linear function

$$\mathbb{C}^{2Nn} \times \Lambda_N \to \Lambda_N \xrightarrow{p} \mathbb{R} \ .$$

Let us begin with the generating function $\mathcal{F}_{\lambda_0}^{(N)} : \mathbb{C}^{2Nn} \times \{\lambda_0\} \to \mathbb{R}$. By construction its critical point x_0 corresponds to a fixed point of the map

$$-\mathrm{id} \circ h_{2N_2} \circ \cdots \circ h_1 \circ t \circ \cdots \circ t : \mathbb{C}^n \to \mathbb{C}^n \ .$$

Let z_0 be such a fixed point. It also corresponds to an intersection point of the conical Lagrangian submanifolds Q and H_{λ_0} in $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N}$. Let us calculate now partial derivatives of $\mathcal{F}_N$ in $\lambda_1, \ldots, \lambda_k$ at the point z_0.

Lemma. $\frac{\partial \mathcal{F}_N}{\partial \lambda_i}(x_0, \lambda_0) = -2N_1 \mathcal{P}_i(z_0)$ *where* $\mathcal{P}_1, \ldots, \mathcal{P}_n : \mathbb{C}^n \to \mathbb{R}$ *are quadratic hamiltonians generating the action on* $\mathbb{C}^n$ *of coordinate 1-parametric subgroups in* T^k.

Indeed: (1) The generating function $\int pdq$ of a *conical* Lagrangian submanifold coincides with the restriction of the quadratic form $pq/2$ to this Lagrangian submanifold.

(2) Its derivative along a linear hamiltonian flow at the point $(p, q) = (0, q_0)$ equals $-\mathcal{P}(0, q_0)$ where $\mathcal{P}$ is the quadratic hamiltonian of the flow.

(3) Choose (p, q)-coordinates on $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N} = T^*\Delta$ in a way that $p = 0$ is an equation of Q, $q =$ const are equations of fibers in $T^*\Delta$. Then $\mathcal{F}_\lambda^{(N)} = \mathcal{Q} - \mathcal{H}_\lambda = -\int pdq|H_\lambda$.

(4) H_λ is obtained from H_{λ_0} by the simultaneous action of T^k on the first $2N_1$ factors $\bar{\mathbb{C}}^n$ in $(\mathbb{C}^n \times \bar{\mathbb{C}}^n)^{2N}$. Hamiltonians of this action are $-\mathcal{P}_i(w_1) - \cdots - \mathcal{P}_i(w_{2N_1})$, $i = 1, \ldots, k$.

Combining (1)-(4) we find

(4.1)
$$\frac{\partial \mathcal{F}_N}{\partial \lambda_i}(x_0, \lambda_0) = -\mathcal{P}_i(tz_0) - \mathcal{P}_i(t^2 z_0) - \cdots = -2N_1 \mathcal{P}_i(z_0) \ .$$

Formula (4.1) should be compared with (2.4),(2.5). The lemma means that

(1) critical points of $\mathcal{F}_N$ correspond to the fixed points z_0 that satisfy the constraints

$$\mathcal{P}_1(z_0) = \cdots = \mathcal{P}_k(z_0) = 0;$$

(2) critical points of the restriction $\mathcal{F}_N|_{p=\mathrm{const}}$ correspond to the fixed points z_0 that satisfy the constraints

$$(\mathcal{P}_1(z_0), \ldots, \mathcal{P}_k(z_0)) \sim p = (p_1, \ldots, p_k).$$

The first system of constraints holds only for $z_0 = 0$ (since some linear combination of $\mathcal{P}_i$ is positive definite due to compactness of M_p, see §1). The second one implies:

$$\exists \mu > 0 : \mu z_0 \in P^{-1}(p)$$

that is, selects fixed points of $(-\,\mathrm{id}) \circ h^1$ on M_p. Similarly to Proposition 2.2 the homogeneity of $\mathcal{F}_\lambda^{(N)}$ completes the proof.

Our construction of generating functions depends on the decomposition

$$h^1 = (-\,\mathrm{id}) \circ h_{2N} \circ \cdots \circ h_1 \ . \tag{4.2}$$

We describe below two moves of the decomposition which change the homotopy type of the corresponding subdivision $\mathcal{S}_N = F^+ \cup F^-$ in a controllable way, provided at least that $F^+ \cap F^-$ is nonsingular (that is, if h^1 does not have non-trivial fixed points).

(1) If the decomposition (4.2) varies in a way that h^1 remains unchanged and all h_i remain small then the homotopy type of the subdivision $\mathcal{S}_N = F^+ \cup F^-$ does not change.

(2) Suppose that the hamiltonian isotopy h^t of $\mathbb{C}^n$ consist of two parts and the first one is a loop (decomposed into $2K$ small pieces, $\mathrm{id} = h_{2K} \circ \cdots \circ h_1$. One may compare the generating function $\mathcal{F}^{(N)}$ of the whole isotopy to the generating functions $\mathcal{F}^{(N-K)}$ and $\mathcal{G}^{(K)}$ of the parts

$$h^1 = (-\,\mathrm{id}) \circ h_{2N} \circ \cdots \circ h_{2K+1}, (-\,\mathrm{id}) \circ h_{2K} \circ \cdots \circ h_1 \ , \quad N > K.$$

We assert that $\mathcal{F}^{(N)}$ *can be deformed to* $\mathcal{F}^{(N-K)} \oplus \mathcal{G}^{(K)}$ *in a way such that the homotopy type of the subdivision* $\mathcal{S}_N = F^+ \cup F^-$ *does not change during the deformation.*

The first statement is obvious. The second one is based on the following deformation Q^ε of the Lagrangian subspace $Q = \{w_1 = z_2, w_2 = z_3, \ldots, w_{2N} = -z_1\}$:

$$\begin{aligned} z_1 + w_{2K} &= \varepsilon(z_{2K+1} - w_{2N}) \\ w_{2N} + z_{2K+1} &= \varepsilon(w_{2K} - z_1) \end{aligned} \tag{4.3}$$

For $\varepsilon = 1$ (4.3) is equivalent to two of the equations for Q: $z_1 = -w_{2K}, w_{2K} = z_{2K+1}$. For $\varepsilon = 0$ we obtain $Q_1 \times Q_2$ instead of Q where Q_1 and Q_2 correspond to cyclic shifts in $(\mathbb{C}^n)^{2K}$ and $(\mathbb{C}^n)^{2(N-K)}$ respectively. Since $H = H_1 \times H_2$

by the very definition, the generating function $\mathcal{F}^{(N)}$ converts into $\mathcal{F}^{(N-K)} \oplus \mathcal{G}^{(K)}$ at $\varepsilon = 0$. In between, for $0 < \varepsilon < 1$, at intersections of $Q^\varepsilon \cap H$ we have $w_{2K} = z_1$ (since the first part of the hamiltonian isotopy forms a loop) and thus (4.3) turns into

$$z_{2K+1} = \frac{1}{\varepsilon} w_{2K}, \qquad z_1 = -\varepsilon w_{2N} \ .$$

This means that the critical rays of $\mathcal{F}_\varepsilon^{(N)}$ are the same as those of $\mathcal{F}^{(N)}$ and implies that the deformation of $\mathcal{F}^{(N)}$ to $\mathcal{F}^{(N-K)} \oplus \mathcal{G}^{(K)}$ does not change the homotopy type of the subdivision $\mathcal{S}_N = F^+ \cup F^-$, at least if $\mathcal{F}^{(N)}$ does not have such critical rays at all.

Now let us consider the family (in the sense of §3)

$$\mathcal{F}_N | \mathcal{S}_N \times \Lambda_N \to \mathbb{R} \ .$$

It was designed in a way that its front $\Phi_N \subset \Lambda_N$ would consist of those $\lambda \in \Lambda_N$ for which $(-\,\mathrm{id}) \circ h^1 \circ \exp(\lambda) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ does not have non-trivial fixed points. Therefore $\Phi_N = \Phi \cap \Lambda_N$ where Φ is the front of a suitable family $\{\mathcal{A}_\lambda\}$ of action functionals (that is, independent on the decomposition). Applying our two moves to decompositions

$$(-\,\mathrm{id}) \circ h_{2N_2} \circ \cdots \circ h_1 \circ t_{2N_1} \circ \cdots \circ t_1, \qquad t_i \in T^k$$

with $t_{2N_1} \circ \cdots \circ t_1 = \exp(\lambda)$ we conclude that

> *$\mathcal{F}_{N+K} | \Lambda_N + m$ with $m \in \mathbb{Z}^k$ deforms fiberwise into $\mathcal{F}_N|_{\Lambda_N} \oplus \mathcal{G}_m^{(K)}$ (where $\mathcal{G}_m^{(K)}$ is the generating function of the "discrete loop" $t \circ \cdots \circ t$ with $t = \exp(m/2K)$) in such a way that the front $\Phi_N \subset \Lambda_N$ of the family remains unchanged during the deformation* (see Figure 4).

Combining this statement with the results of §3 (see Corollary 3.2 and Remarks 3.4 and 4.1.3) and assuming that the cubes Λ_N are chosen transversal to the (periodic, zero measure) front $\Phi \subset \mathbb{R}^k$ we obtain the following

Proposition 4.4. *Let $m \in \mathbb{Z}^k$ and $\Lambda_N + m \subset \Lambda_{N+K}$. Let $\mathcal{G}_m^{(K)} : \mathbb{C}^{2Kn} \to \mathbb{R}$ be the Hermitian generating form of the "twisted discrete loop" $(-\mathrm{id}) \circ t \circ \cdots \circ t$ (2K times), $t = \exp(m/2K) \in T^k$. Let $\mathcal{F}_N$ and $\mathcal{F}_{N+K}$ be the generating families of the same exhausting sequence. Then the restricted family*

$$\mathcal{F}_{N+K}|(\Lambda_N + m)$$

is fiberwise homotopy equivalent to the fiberwise suspension

$$\mathcal{F}_N \oplus_{\Lambda_N} \mathcal{G}_m^{(K)}$$

in the sense that they have the same fronts and for all $\Gamma \subset \Lambda_N$ transversal to the front $\Phi_N \subset \Lambda_N$ there is a simultaneous T^k-equivariant homotopy equivalence of the subdivisions $\mathcal{S}_N \times \Gamma = F^+ \cup F^-$ determined by

$$\mathcal{F}_{N+K}|_{\Gamma+m} \qquad \textit{and} \qquad \mathcal{F}_N|_\Gamma \oplus_\Gamma \mathcal{G}_m^{(K)}\ .$$

This homotopy equivalence will serve us as a basis for constructing a "semi-infinite limit" of our finite-dimensional approximations and at the same time for introducing a $\mathbb{Z}^k$-action (Novikov's structure) in this limit.

4.5 Examples: Quadratic generating families. . We consider here "generating families" with $N = N_1$ (which actually generate nothing since $N_2 = 0$).

(1) $k = n = 1$. The torus T^1 acts by $\exp(2\pi\lambda)$ on $\mathbb{C}^1$. A straightforward computation shows that $\mathcal{Q} : \mathbb{C}^{2N} \to \mathbb{R}$ is an Hermitian form with the matrix $i(1-U)/(1+U)$ which is a Cayley transform of the unitary matrix of the cyclic shift

$$U = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ -1 & & & 0 \end{bmatrix}$$

The spectrum of $\mathcal{Q}$ is $\tan(\frac{\pi}{2N}\ell)$ where $-N < \ell < N$ is *odd integer*. The matrix of $\mathcal{H}$ is Cayley's transform of the *scalar* matrix $t = \exp(\pi\lambda/N)$, that is, $\tan(\frac{\pi\lambda}{2N})$. We find that the maximal cube Λ_N is the interval $-N < \lambda < N$, and the *signature* of the Hermitian generating family $\mathcal{F}_N = \mathcal{Q} - \mathcal{H}$ depends linearly on λ (as displayed on Figure 5a),

$$\text{(4.4)} \qquad \operatorname{sign} \mathcal{F}_\lambda^{(N)} = -2\ell \quad \textit{for} \quad \ell - \tfrac{1}{2} < \lambda < \ell + \tfrac{1}{2}, \quad \ell \in \mathbb{Z}.$$

(2) $k = n > 1$. It is the direct sum of n copies of example 1. The representation of the torus T^n in $\mathbb{C}^{2nN}$ decomposes by its n coordinate characters into n isotypical $2N$-dimensional components. The *equivariant signature* of an invariant Hermitian form (by definition it is the collection of signatures of its restrictions to isotypical components of the representation) is in this case simply a vector sign $= (s_1, \dots, s_n)$ of n even integers. Such a vector-function $\lambda \mapsto \operatorname{sign} \mathcal{F}_\lambda^{(N)}$ is the direct sum of n copies of (4.4) (see Figure 5b for $k = n = 2$).

(3) $k < n$. The generating family in this case is obtained by restriction of the generating family of example 2 to the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ in the parameter space.

(4) The Hermitian form $\mathcal{G}_m^{(K)}$ from Proposition 4.4 is just the member of the family of example 2 with $\lambda = m \in \mathbb{Z}^k \subset \mathbb{R}^n$, and its T^n-equivariant signature can also be found on the diagram 5b.

§5. Equivariant cohomology

We formulate here following [Hs] some facts about torus-equivariant cohomology and then describe the topological properties of generating families.

Let X be a topological space provided with an action of a compact Lie group G. Equivariant cohomology $H_G^*(X)$ is defined as the usual singular cohomology $H^*(X_G)$ of the homotopy quotient

$$X_G = (X \times EG)/G \ .$$

Here $EG \to BG$ is the universal principal G-bundle over the classifying space BG. The canonical projection

$$(X \times EG)/G \to EG/G$$

provides $H^*_G(X)$ with a module structure over the characteristic class algebra $H^*(BG)$ playing the role of the coefficient ring $H^*_G(pt)$ in equivariant cohomology theory.

In this paper we need only equivariant cohomology of torus actions. In this case the coefficient ring $H^*(BT^n)$ is a polynomial algebra in n variables $u = (u_1, \ldots, u_n)$ of degree 2 since the universal bundle of the circle T^1 is the Hopf bundle $S^\infty \to \mathbb{C}P^\infty$. We will assume all singular cohomology to be with complex (rather than integer) coefficients so that $H^*(Y) \overset{def}{=} H^*(Y, \mathbb{C})$ and thus

$$H^*_{T^n}(pt) = \mathbb{C}[u_1, \ldots, u_n] \ .$$

We list below some properties of T^n-equivariant cohomology (see [Hs]).

1. Let X be a T^n-space, $T^k \subset T^n$ a subgroup. Then the homotopy quotients X_{T^n} and X_{T^k} form the bundle

$$X_{T^k} \xrightarrow{T^n/T^k} X_{T^n}$$

since ET^n can be considered as ET^k if provided with the subgroup action. The spectral sequence of this bundle has E_2-term

$$E_2^{p,q} = H^p_{T^n}(X) \otimes H^q(T^n/T^k) \tag{5.1}$$

and converges to $H^*_{T^k}(X)$. The algebra $H^*(T^n/T^k)$ is the exterior algebra with $n-k$ generators of degree 1. Their transgression images in $H^2(BT^n)$ generate in $\mathbb{C}[u_1, \ldots, u_n]$ the ideal I of the subspace $\mathbb{R}^k = \text{Lie } T^k \subset \text{Lie } T^n = \mathbb{R}^n$. It will be important for us that the E_2-term (5.1) is nothing but *the Koszul complex* (see [GrH]) *of these $n-k$ degree 2 elements in the algebra $H^*_{T^n}(X)$.*

2. *Kunneth spectral sequence* for equivariant cohomology is obtained when one considers the product $X \times Y$ of two G-spaces as a $G\times G$-space provided with the action of the diagonal subgroup $G \subset G \times G$. It has the E_2-term

$$E_2^{*,q} = \text{Tor}^q_{H^*(BG\times BG)}(H^*(BG) \otimes H^*(X_G \times Y_G)) \ .$$

In the special case when $H^*_G(Y)$ *is a free* $H^*_G(pt)$-module the spectral sequence reduces to one line $q = 0$ *provided at least that* G *is a torus.* Indeed, a resolution of $H^*(X_G \times Y_G)$ over $H^*(BG \times BG) = \mathbb{C}[u, v]$ is just the tensor product of a resolution of $H^*(X_G)$ over $\mathbb{C}[u]$ and a free $\mathbb{C}[v]$-module $H^*(Y_G)$ while $\mathbb{C}[u, v]$-module $H^*(BG)$ is $\mathbb{C}[u, v]/(u_i = v_i)$. This results in $\mathrm{Tor}^q = 0$ for $q > 0$. We conclude that

$$H^*_G(X \times Y) = H^*_G(X) \otimes H^*_G(Y)$$

if G is a torus and $H^*_G(Y)$ is free.

Our objective now is to study equivariant cohomology of some spaces related to invariant homogeneous functions and their families.

We begin with a geometric fact well known in singularity theory (cf. [AVG], v.2). Let $\mathcal{F} : \mathbb{C}^M \to \mathbb{R}$ be a homogeneous degree 2 function, $\hat{\mathcal{F}} : \mathbb{C}^{M+1} \to \mathbb{R}$ be its suspension:

$$\hat{\mathcal{F}}(x, z) = \mathcal{F}(x) + |z|^2 \ .$$

We want to compare homotopy types of the corresponding ray spaces $\hat{F}_\pm$ and $F_\pm$.

Proposition 5.1. $\hat{F}_+ \approx F_+ * S^1$, $\hat{F}_- \approx F_-$ *where* $*$ *means join of topological spaces and* $S^1 = \{z \in \mathbb{C} |\, |z| = 1\}$*. If* $\mathcal{F}$ *is invariant relative to a* T^1*-action on* $\mathbb{C}^M$ *then the homotopy equivalences* $\approx$ *can be done equivariant with respect to the diagonal action on* $\mathbb{C}^{M+1}$*:* $t(x, z) = (tx, tz)$*. If* $\mathcal{F}$ *depends on additional parameters then the homotopy depends continuously on them.*

Proof. Let us begin with the case when z is a *real* variable. In Figure 6 the north semi-sphere D of rays in $\mathbb{C}^M \oplus \mathbb{R}$ is shown. It splits into $\hat{F}_+$ and $\hat{F}_-$. Every meridional semicircle (through the north pole $P = (0, 1)$) meets $\hat{F}_+$ along an arc about the pole. The meridional contraction from the pole towards the equatorial sphere ∂D contracts $\hat{F}_-$ to F_-. On the other hand, the same contraction transforms the pair $(\hat{F}_+ \cap D, F_+)$ to (D, F_+). Obviously the pair (D, F_+) is homotopy equivalent to (CF_+, F_+) where CF_+ is the cone (from P) over its bottom $F_+ \subset \partial D$.

Now, if z is a complex variable, we have a polar circle S^1, and the ray sphere in $\mathbb{C}^{M+1}$ is the union of identical semi-spheres D, parametrized by S^1, with common equator ∂D. From what was said above we conclude that $\hat{F}_+$ is homotopy equivalent to

$$S^1 \times CF_+/((z,x) \sim (z',x) \text{ for } x \in F_+) ,$$

that is, to $S^1 * F_+$.

Since all the homotopies above are carried out in a canonical way, they respect group actions and parametric dependence. ■

It is time now to disclose the sort of equivariant cohomology that we intend to use in the proof of our main theorem. Let us consider the family

$$\mathcal{F}_N : \mathbb{C}^M \times \Lambda_N \to \mathbb{R} , \qquad M = 2nN ,$$

of homogeneous T^k-invariant functions parametrized by the cube Λ_N. Let

$$\mathcal{S}_N \times \Lambda_N = F^+ \cup F^-$$

be the decomposition of the ray space into its positive and negative parts (with respect to $\mathcal{F}_N$), and

$$\partial F^{\pm} = F^{\pm} \cap (\mathcal{S}_N \times \partial\Lambda_N)$$

—the parts of $F^{\pm}$ situated over the boundary of the cube Λ_N. We intend to deal with the *relative T^k-equivariant cohomology* (with complex coefficients)

$$H^*_{T^k}(F^-, \partial F^-) \ .$$

In order to figure out the algebraic structure of this module it is convenient to introduce the pair (CF^-, F^-), where $CF^- \subset \mathbb{C}^M \times \Lambda_N$ means *the fiberwise cone* over F^-, that is, the cone of the projection map $F^- \to \Lambda_N$. It is natural to call the quotient space CF^-/F^- *the fiberwise suspension* and denote it $\mathbf{\Sigma} F^-$ referring to the quotient point F^- as the distinguished one.

From the long exact sequence of the pair (CF^-, F^-)

$$\to H^{*-1}_{T^k}(F^-, \partial F^-) \to H^*_{T^k}(\mathbf{\Sigma} F^-, \mathbf{\Sigma}\partial F^-) \to$$

$$\to H^*_{T^k}(CF^-, C\partial F^-) \to$$

one extracts the short exact segments

$$(5.2) \quad 0 \to \tilde{H}^{*-1}_{T^k}(F^-, \partial F^-) \to H^*_{T^k}(\mathbf{\Sigma} F^-, \mathbf{\Sigma}\partial F^-) \to J_F \to 0$$

where J_F is the *kernel* of the homomorphism

$$H^*_{T^k}(\Lambda_N, \partial\Lambda_N) \to H^*_{T^k}(F^-, \partial F^-) \ ,$$

and $\tilde{H}^*$ denotes its *cokernel.* We will call this cokernel the *reduced* equivariant cohomology. Since $\Lambda_N/\partial\Lambda_N$ is homeomorphic to a sphere with the trivial torus action we find $H^*_{T^k}(\Lambda_N, \partial\Lambda_N)$ to be *a free $H^*_{T^k}(pt)$-module of rank 1 generated by the fundamental cocycle of the sphere, and J_F is a submodule in this module*. The exact segments (5.2) reduce computation of $H^*_{T^k}(F^-, \partial F^-)$ to that of J_F and $\tilde{H}^*_{T^k}(F^-, \partial F^-)$. We state below how the suspension of Proposition 4.4 effects these data.

Proposition 5.2. *Let $\hat{\mathcal{F}}_{N+K} = \mathcal{F}_N \oplus_{\Lambda_N} \mathcal{G}_m^{(K)}$ be the fiberwise direct sum of the family $\mathcal{F}_N$ with the Hermitian form $\mathcal{G}_m^{(K)} : \mathbb{C}^{2Kn} \to \mathbb{R}$ of Proposition 4.4 and $\hat{F}^-, F^-$ denote the corresponding ray spaces. Then*

$$\tilde{H}^*_{T^k}(\hat{F}^-, \partial\hat{F}^-) = \tilde{H}^*_{T^k}(F^-, \partial F^-) \otimes I^{K+m} \ , \tag{5.3}$$

$$J_{\hat{F}} = J_F \otimes I^{K+m} \ ,$$

where $\otimes$ means the tensor product of $H^*_{T^k}(pt)$-modules, and I^{K+m} is the principal ideal in $H^*_{T^k}(pt)$ generated by the monomial $u^{K+m} = u_1^{K+m_1} \cdot u_2^{K+m_2} \cdot \ldots \cdot u_n^{K+m_n}$ (recall that $H^*_{T^k}(pt)$ is a quotient of $H^*_{T^n}(pt) = \mathbb{C}[u_1, \ldots, u_n]$).

Proof. It consists of the following three kinds of arguments.

(1) An equivariant decomposition of the Hermitian form $\mathcal{G}_m^{(k)}$ into the direct sum of one-dimensional forms and multiple application of Proposition 5.1 shows that $\hat{F}^-$ is *the fiberwise join* $F^- *_{\Lambda_N} G^-$ of F^- with the negative ray space G^- of the Hermitian form $\mathcal{G}_m^{(K)}$.

(2) Equivariant cohomology of joins. Let $\mathbb{Y}$ be a family of T^k-spaces parametrized by Λ, X be another T^k-space, and $X *_\Lambda \mathbb{Y}$ be their fiberwise join. The fiberwise suspension operation $\mathbf{\Sigma}$ transforms functorially fiberwise joins into *tensor products*:

$$\mathbf{\Sigma}(X *_\Lambda \mathbb{Y}) \simeq \Sigma X \otimes \mathbf{\Sigma}\mathbb{Y}$$

where $\otimes$ is the product operation in the category of punctured spaces,

$$(A, pt) \otimes (B, pt) = A \times B/(A \times pt) \cup (pt \times B) \ .$$

Now Proposition 5.2 reduces to the (relative) equivariant Kunneth formula provided that $H^*_{T^k}(\Sigma G^-, pt)$ is a free $H^*_{T^k}(pt)$-module (see point 2 above).

(3) The following computation of $H^*_{T^k}(\Sigma G^-, pt)$ completes the proof.

Proposition 5.3. $J_G = I^{K+m}, \tilde{H}^*_{T^k}(G^-) = 0$.

We consider here $\mathcal{G}_m^{(K)}$ formally as a "family" parametrized by a point so that $J_G \subset H^*_{T^k}(pt)$.

Proof:. inductive application of Proposition 5.1. Let us begin with the S^1-invariant form $a|z|^2$ on $\mathbb{C}^1$. Then

$$G^- = \begin{cases} S^1 & a \leq 0 \\ \varnothing & a > 0\ , \end{cases}$$

and a simple straightforward computation gives

$$H^*_{S^1}(\Sigma G^-, pt) = \begin{cases} u\mathbb{C}[u] & a \leq 0 \\ \mathbb{C}[u] & a > 0\ . \end{cases}$$

The same answer is still valid for the "multiple" circle action $(e^{i\phi}, z) \mapsto e^{in\phi}z$, $n \in \mathbb{Z}$ (it is essential here that we use rational — or even complex — coefficients instead of integers).

Now suppose that T^k acts on $\mathbb{C}^1$ through its character $\chi : T^k \to S^1$ and denote u an equation of kerχ in Lie T^k (identifying $H^*_{T^k}(pt)$ with $\mathbb{C}[\text{Lie } T^k]$). Then

$$H^*_{T^k}(\Sigma G^-, pt) = \begin{cases} uH^*_{T^k}(pt) & a \leq 0 \\ H^*_{T^k}(pt) & a > 0\ . \end{cases}$$

Let us now consider the generating Hermitian form $\mathcal{G}_m^{(K)}$ of the "discrete loop" $-id \circ t \circ \cdots \circ t$ ($2K$ times), where $t = \exp(\pi i m/K)$ is a hamiltonian transformation of $\mathbb{C}^1$. Then $\mathcal{G}_m^{(K)}$ has $K - m$ positive and $K + m$ negative squares (cf. Example 4.5). Applying Proposition 5.1 together with the equivariant Kunneth formula we find

$$H^*_{T^k}(\Sigma G^-, pt) = u^{K+m} H^*_{T^k}(pt)\ .$$

In the end we notice that $\mathcal{G}_m^{(k)}$ in Proposition 5.3 corresponds to the transformation

$$t = \text{diag}(e^{\pi i m_1/K}, \ldots, e^{\pi i m_n/K})$$

on $\mathbb{C}^n$ and is the direct sum of n $2K$-dimensional Hermitian forms considered above. Applying Proposition 5.1 and the Kunneth formula once again we conclude that

$$H^*_{T^k}(\Sigma G^-, pt) = u_1^{K+m_1} \dots u_n^{K+m_n} H^*_{T^k}(pt) \ ,$$

where $u_1, \dots, u_n$ are images in $H^*_{T^k}(pt)$ of generators in $H^*_{T^n}(pt)$.■

Our last step in this section is an asymptotic computation of the relative equivariant cohomology $H^*_{T^k}(F^-, \partial F^-)$ for the *quadratic* generating family $\mathcal{F}_N : \mathbb{C}^{2nN} \times \Lambda_N \to \mathbb{R}$ of Example 4.5 *in the limit* $N \to \infty$. To my taste it is the most elegant point in the proof of our main theorem.

We begin with the case $k = n$, that is, $\{\Lambda_N\}$ is a sequence of cubes exhausting the parameter space $\mathbb{R}^n = \text{Lie } T^n$. Our objective is to restrict the quadratic families $\mathcal{F}_N$ to *a convex compact r-dimensional polyhedron* $\Gamma \subset \mathbb{R}^n$ and calculate

$$\mathcal{H}^*_{T^n}(\Gamma) = \lim_{N \to \infty} u^{-N} \cdot H^{*+2N}_{T^n}(F_N^-|_\Gamma, F_N^-|_{\partial\Gamma}) \ .$$

(We will see in the next section how such a limit corresponds to some semi-infinite cohomology.)

The answer is that the reduced part of this cohomology is trivial ($\tilde{\mathcal{H}}^*_\Gamma = 0$) and the corresponding "kernel" module J_Γ can be easily described in terms of Newton diagrams in the same space $\mathbb{R}^n$. This means that we should interpret the lattice $\mathbb{Z}^n$ in the parameter (!) space $\mathbb{R}^n$ as the lattice of exponents of monomials $u^\ell = u_1^{\ell_1} \dots u_n^{\ell_n}$. To be more precise, *we introduce in* $\mathbb{R}^n$ *new coordinates* $\mu_1 = \lambda_1 - \frac{1}{2}, \dots, \mu_n = \lambda_n - \frac{1}{2}$, *associate an integer point* $\mu = \ell$ *with the monomial* $u^\ell \in \mathbb{C}[u, u^{-1}]$ *and attach some open Newton diagram* Δ_Γ *to the polyhedron* Γ:

$$\Delta_\Gamma = \{\mu \in \mathbb{R}^n | \exists \gamma \in \Gamma : \mu_i > \gamma_i, i = 1, \dots, n\} = \Gamma + \mathbb{R}^n_+$$

(see Figure 7a).

Proposition 5.4.

$$\mathcal{H}^*_{T^n}(\Gamma) = H^r(\Gamma, \partial\Gamma) \otimes_\mathbb{C} \mathbb{C}[u, u^{-1}]/J_\Gamma \ ,$$

where J_Γ is the $\mathbb{C}[u]$-submodule in $\mathbb{C}[u,u^{-1}]$ generated by the monomials u^ℓ with $\ell \in \Delta_\Gamma$.

Proof. The idea is to apply Leray spectral sequences [GrH]. The Leray spectral sequence of a map $X \xrightarrow{\pi} \Lambda$ converges to the cohomology of the "total space", $H^*(X)$, and has

$$E_2^{p,q} = H^p(\Lambda, \mathcal{J}^q)$$

where $\mathcal{J}^*$ is the local coefficient sheaf whose fiber at $\lambda \in \Lambda$ is $\mathcal{J}^*_\lambda = H^*(\pi^{-1}(\lambda))$. The same construction applies to equivariant cohomology provided that the group action on X is fiberwise.

In our case $\Lambda = \Gamma \subset \mathbb{R}^n$, $X = F^-$, and the equivariant cohomology of fibers has just been described in Proposition 5.3:

$$\mathcal{J}_\lambda = \mathbb{C}[u]/(u^{N+m}) \ , \qquad m = [\lambda + \tfrac{1}{2}] = [\mu + 1] \ ,$$

(see Examples 4.5.1, 4.5.2). We will find below

$$E_2^{p,*} = H^p(\Gamma, \partial\Gamma; \mathcal{J}^*)$$

and observe that it is non-trivial only for $p = r$. This will mean in particular that $E_2^{*,*} = E_\infty^{*,*} = H^*_{T^n}(F^-|_\Gamma, F^-|_{\partial\Gamma})$.

We will carry out our computation of $E_2^{*,*}$ in the limit of large N. This makes convenient the following "re-grading" of the sheaf $\mathcal{J}^*$:

$$\hat{\mathcal{J}}^*_\lambda = \lim_{N\to\infty} u^{-N}\mathbb{C}[u]/(u^m) = \mathbb{C}[u,u^{-1}]/\mathbb{C}[u]\cdot u^m \ .$$

The idea of our computation of $H^*(\Gamma, \partial\Gamma; \hat{\mathcal{J}}^*_\lambda)$ is to decompose the constant sheaf $\mathbb{C}[u,u^{-1}]$ into one-dimensional monomial components $\mathbb{C}\cdot u^\ell$ and apply the following pretty obvious general lemma.

Lemma. *Let $\mathbb{C}$ be a constant sheaf on a topological space X, $\mathbb{I}_Y$ be the subsheaf vanishing on the closed subspace $Y \subset X$, $\mathbb{C}/\mathbb{I}_Y$ be the quotient sheaf supported at Y. Then*

$$H^*(X, \mathbb{I}_Y) = H^*(X, Y) \ , \qquad H^*(X, \mathbb{C}/\mathbb{I}_Y) = H^*(Y) \ .$$

The monomial $u^\ell \in \mathbb{C}[u, u^{-1}]$ is contained in the $\mathbb{C}[u]$-submodule (u^m) only iff $\ell \geq m$. This event is supported in the open "last orthant" $\mu_j < \ell_j$, $j = 1, \ldots, n$. Let Y be its complement in $\mathbb{R}^n$. Applying the lemma we find (Figure 7b)

$$H^*(\Gamma, \partial\Gamma; \mathbb{C}/\mathbb{I}_Y|_\Gamma) = \begin{cases} \mathbb{C} & * = r \text{ and } \Gamma \subset Y \\ 0 & \text{otherwise} \end{cases}$$

since $\Gamma/\partial\Gamma$ is an r-dimensional sphere and $\Gamma \cap Y$ contracts to $\partial\Gamma$ if $\Gamma \not\subset Y$. Taking the direct sum over all the monomials u^ℓ we get

$$H^*(\Gamma, \partial\Gamma; \hat{\mathcal{J}}^*) = H^r(\Gamma, \partial\Gamma) \otimes_{\mathbb{C}} \mathbb{C}[u, u^{-1}]/J_\Gamma$$

where the $\mathbb{C}[u]$-submodule J_Γ is spanned by those monomials u^ℓ whose "last orthant" $\{\mu|\ell+\mathbb{R}^n_-\}$ meets Γ. Or, in other words, (Figure 7a)

$$J_\Gamma = \langle u^\ell | \ell \in \mathbb{Z}^n : \ell \in \Gamma + \mathbb{R}^n_+ \rangle \ .$$

Now let us consider the case where Γ_N is a sequence of, say, $k-1$-dimensional cubes exhausting a $(k-1)$-*dimensional affine subspace* Γ *in* $\mathbb{R}^n$ *parallel to the subspace* $\mathbb{R}^k \cap p^{-1}(0) \subset \mathbb{R}^k \subset \mathbb{R}^n$. Then $J_{\Gamma_N} \subset J_{\Gamma_{N+1}} \subset \dots$ and the union is the submodule $J_\Gamma \subset \mathbb{C}[u, u^{-1}]$ spanned by the monomials u^ℓ supported at the Newton diagram $\Delta_\Gamma = \Gamma + \mathbb{R}^n_+$. The submodule J_Γ should be compared to

$$J_r = \langle u^\ell, \ell \in \mathbb{Z}^k \subset \mathbb{Z}^n | \langle p, \ell \rangle \geq r \rangle_{\mathbb{C}[u]}$$

considered in the end of §1, and one concludes that they are basically the same (up to a shift of the origin in the lattice $\mathbb{Z}^n$). To be more precise, J_r ($r \in \mathbb{R}$) and J_Γ (Γ is parallel to $\mathbb{R}^k \cap p^{-1}(0)$) majorate each other:

$$\begin{aligned} &\forall \Gamma \, \exists r_+ > r_- : J_{r_+} \subset J_\Gamma \subset J_{r^-}, \\ &\forall r \, \exists \Gamma_+, \Gamma_- : J_{\Gamma_+} \subset J_r \subset J_{\Gamma_-}. \end{aligned} \tag{5.5}$$

Let us denote $\mathcal{J}_\Gamma$ (resp. $\mathcal{J}_r$) projections of J_Γ (resp. J_r) to $\mathcal{R} = \mathbb{C}[u, u^{-1}]/I \cdot \mathbb{C}[u, u^{-1}]$, where I is the ideal of $\mathbb{C}^k \subset \mathbb{C}^n$ in $\mathbb{C}[u]$. The same inclusions (5.5) hold for $\mathcal{J}_\Gamma, \mathcal{J}_r$ of course.

Corollary 5.5 ($k < n$)**.**

$$\lim_{N \to \infty} \mathcal{H}^*_{T^k}(\Gamma_N) = H^{k-1}(\Gamma, \infty) \underset{\mathbb{C}}{\otimes} \mathcal{R}/\mathcal{J}_\Gamma.$$

Proof. T^k-equivariant cohomology is related to T^n-equivariant cohomology by means of Serre's spectral sequence whose E_2-term is the Koszul complex (5.1). For a regular linear function $p : \mathbb{R}^k \to \mathbb{R}$ the $n-k$ equations of $\mathbb{R}^k$ in $\mathbb{R}^n$ form a regular sequence in the algebra

$$\mathbb{C}[u, u^{-1}]/\mathbb{C}[u, u^{-1}] J_\Gamma$$

since it is true for J_r instead of J_Γ (as it was shown in §1) and due to inclusions (5.5). Therefore the Koszul complex has only 0-dimensional (in our grading) homology

$$E_2^{*,0} = (\mathbb{C}[u,u^{-1}]/(\mathbb{C}[u,u^{-1}]I + J_\Gamma)) \underset{\mathbb{C}}{\otimes} H^{k-1}(\Gamma,\infty),$$

and the spectral sequence degenerates, that is $E_2 = E_\infty$.

§6. Semi-infinite cohomology

We combine here Propositions 4.4 and 5.2 in order to define semi-infinite cohomology related with our generating families, describe its Morse-theoretic properties and prove Theorem 1.1.

Throughout this section $p : \mathbb{R}^k \to \mathbb{R}$ is a regular primitive integer value of the momentum map of the T^k-action in $\mathbb{C}^n$.

Let $\mathcal{F}_N : \mathcal{S}_N \times \Lambda_N \to \mathbb{R}$ be generating families (of the hamiltonian transformation $h^1 : M_p \to M_p$) constructed in §4 so that $\Lambda_N \subset \Lambda_{N'} \subset \cdots \subset \mathbb{R}^k$ form an exhausting sequence of cubes. Let

$$\Gamma_N(\nu) = \Lambda_N \cap p^{-1}(\nu)$$

be their intersection with hyperplane levels of the linear function p. Let $F_N^-(\nu), \partial F_N^-(\nu)$ denote negative ray spaces of the families $\mathcal{F}_N|_{\Gamma_N(\nu)}$, $\mathcal{F}_N|_{\partial\Gamma_N(\nu)}$. For $N < N'$ we have natural homomorphisms

(6.1)
$$H^*_{T^k}(F^-_{N'}(\nu)|_{\Gamma_N(\nu)}, F^-_{N'}(\nu)|_{\partial\Gamma_N(\nu)}) \to H^*_{T^k}(F^-_{N'}(\nu), \partial F^-_{N'}(\nu)).$$

For generic ν the hyperplane $p^{-1}(\nu)$ is transversal to the front of the generating families $\mathcal{F}_N$ and in accordance with Proposition 4.4 there is an equivariant homotopy equivalence of the families

$$\mathcal{F}_{N'}|\Gamma_N(\nu) \text{ and } (\mathcal{F}_N \oplus \mathcal{G}_0^K)|\Gamma_N(\nu), \quad K = N' - N.$$

Together with isomorphisms (5.3), (5.4) of Proposition 4.4 (applied to the families restricted to $\Gamma_N(\nu)$) this gives rise to the raising dimensions monomorphisms (6.2), (6.3):

(6.2)

$$H^{*+2N}_{T^k}(F^-_N(\nu), \partial F^-_N(\nu)) \to H^{*+2N'}_{T^k}(F^-_{N'}(\nu)|_{\Gamma_N(\nu)}, F^-_{N'}(\nu)|_{\partial\Gamma_N(\nu)})$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

(6.3)

$$H^{*+2N}_{T^k}(\Gamma_N(\nu), \partial\Gamma_N(\nu)) \xrightarrow{u^K\times} H^{*+2N'}_{T^k}(\Gamma_N(\nu), \partial\Gamma_N(\nu)).$$

Combining (6.1) and (6.2) we obtain homomorphisms

$$H^{*+2N}_{T^k}(F^-_N(\nu), \partial F^-_N(\nu)) \to H^{*+2N'}_{T^k}(F^-_{N'}(\nu), \partial F^-_{N'}(\nu)). \tag{6.4}$$

Definition 6.1. We define *semi-infinite equivariant cohomology*

$$\mathcal{H}^*_{T^k}(F^-(\nu)) = \lim_{N\to\infty} H^{*+2N}_{T^k}(F^-_N(\nu), \partial F^-_N(\nu))$$

as the direct limit over the directed system (6.4).

Let us list some algebraic properties of $\mathcal{H}^*_{T^k}$.

1. Spaces $\mathcal{H}^*_{T^k}(F^-(\nu))$ descend the structure of graded $H^*_{T^k}(pt)$-modules of ordinary equivariant cohomology.

2. The direct limit $\mathcal{H}^*_{T^k}(\Gamma(\nu))$ of

$$H^{*+2N}_{T^k}(\Gamma_N(\nu), \partial\Gamma_N(\nu)) \xrightarrow{u^K\times} H^{*+2N'}_{T^k}(\Gamma_{N'}(\nu), \partial\Gamma_{N'}(\nu))$$

is isomorphic to $H^*(\mathbb{R}^{k-1}, \infty)\underset{\mathbb{C}}{\otimes}\mathcal{R}$, where

$$\mathcal{R} = \mathbb{C}[u, u^{-1}]/\mathbb{C}[u, u^{-1}]I,$$

(I is the ideal of $\mathbb{C}^k \subset \mathbb{C}^n$ in $\mathbb{C}[u]$) and $\mathbb{R}^{k-1} = \Gamma(\nu) = p^{-1}(\nu)$. The diagram (6.2-6.3) gives rise to the "augmentation homomorphisms" of $\mathbb{C}[u]$-modules

$$\mathcal{H}^*_{T^k}(\Gamma(\nu)) \to \mathcal{H}^*_{T^k}(F^-(\nu)). \tag{6.5}$$

We denote its kernel $\mathcal{J}^*(F^-(\nu))$ and cokernel $\tilde{\mathcal{H}}^*_{T^k}(F^-(\nu))$ and call the latter the *reduced* semi-infinite cohomology.

3. Our computation in the end of §5 shows that for the quadratic generating families of Example 4.5.3 (let us denote them here $\mathcal{G}_N$) $\tilde{\mathcal{H}}^*(\mathcal{G}^-(\nu)) = 0$ and

$$\mathcal{J}^*(G^-(\nu)) = \mathcal{J}_{\Gamma(\nu)} \quad \text{(in notations of §5)}. \tag{6.6}$$

In the very bottom of all our constructions there is a homogeneous degree 2 hamiltonian in $\mathbb{C}^n$. Such a hamiltonian $\mathcal{H}$ can be bounded by quadratic hamiltonians, $\mathcal{H}_- < \mathcal{H} < \mathcal{H}_+$. It is easy to see that this leads to an estimate of ray spaces

$$G_N^-(\nu_-) \subset F_N^-(\nu) \subset G_N^-(\nu_+)$$

for some $\nu_- < \nu < \nu_+$ and to homomorphisms

$$\mathcal{H}^*_{T^k}(G^-(\nu_+)) \to \mathcal{H}^*_{T^k}(F^-(\nu)) \to \mathcal{H}^*_{T^k}(G^-(\nu_-)).$$

Together with (6.5), (6.6) and (5.5) this gives rise to the estimate

$$\mathcal{J}_{r_+} \subset \mathcal{J}^*(F^-(\nu)) \subset \mathcal{J}_{r_-} \tag{6.7}$$

(where $\mathcal{J}_{r_\pm}$ are images of $J_{r_\pm}$ of (5.5) in $\mathcal{R}$, see §1). In particular $\mathcal{J}^*(F^-(\nu))$ is a nontrivial submodule in $\mathcal{R}$.

4. Applying Proposition 4.4 and 5.2 in a manner similar to our definition of semi-infinite cohomology but with non-zero $m \in \mathbb{Z}^k$ we obtain isomorphisms

$$U_m : \mathcal{H}^*_{T^k}(F^-(\nu)) \xrightarrow{\simeq} \mathcal{H}^*_{T^k}(F^-(\nu + p(m))$$

which form lattice's action: $U_m \circ U_{m'} = U_{m+m'}$. This is nothing but Novikov's action of the fundamental group $H_2(M_p, \mathbb{Z})$ of the loop space $\mathcal{L}M_p$. Notice that the sublattice $\mathbb{Z}^{k-1} = p^{-1}(0) \cap \mathbb{Z}^k$ acts on the semi-infinite cohomology space $\mathcal{H}^*_{T^k}(F^-(\nu))$ itself providing it with structure of $\mathbb{C}[\mathbb{Z}^{k-1}]$-module.

The action of U_m on $\mathcal{R}$ is just the multiplication by $u^m = u_1^{m_1} \dots u_n^{m_n}$. Therefore *multiplication by u^m with $m \in \mathbb{Z}^k \subset \mathbb{Z}^n$ is an isomorphism*

$$U^m : \mathcal{J}^*(F^-(\nu)) \xrightarrow{\simeq} \mathcal{J}^*(F^-(\nu + p(m)). \tag{6.8}$$

In particular $\mathcal{J}^*(F^-(\nu))$ is invariant with respect to such multiplication if $p(m) = 0$.

Now let us consider the action functional $\hat{p} : A \to \mathbb{R}$ of §2. According to Proposition 2.2 its (lattices of) critical orbits correspond to fixed points of the hamiltonian transformation $h^1 : M_p \to M_p$.

Proposition 6.2. *Suppose that the function $\hat{p}$ does not have critical values on the closed segment $[\nu_0, \nu_1] \subset \mathbb{R}$. Then $\mathcal{H}^*_{T^k}(F_-(\nu_0)) \simeq \mathcal{H}^*(F_-(\nu_1))$.*

Proposition 6.3. *Suppose that the function $\hat{p}$ has only one critical value within the segment $[\nu_0, \nu_1]$ and that all the critical T^k-orbitst on this level are isolated. Let $v \in H^*_{T^k}(pt)$ be a positive degree element and $q_0 \in \mathcal{H}^*_{T^k}(F^-(\nu_0))$, $q_1 \in \mathcal{H}^*_{T^k}(F^-(\nu_1))$ be images of the same $q \in \mathcal{R}$ under the homomorphism* (6.5). *Then $q_0 = 0$ implies $vq_1 = 0$.*

Proof. Proof of Proposition 4.1 is based of course on Proposition 4.3 and "gradient flow" deformations between $F^-_N(\nu_0)$ and $F^-_N(\nu_1)$. But we should prevent their "topology" from "flowing out" through their boundaries $\partial F^-_N(\nu)$.

Lemma. *If ν_* is a regular value of $\hat{p}$ then there exists a nonzero segment $[\nu_* - \varepsilon, \nu_* + \varepsilon]$ and an exhausting sequence of cubes $\Lambda_N \subset \Lambda_{N+1} \subset \cdots \subset \mathbb{R}^k$ such that all $\Gamma_N(\nu), \partial\Gamma_N(\nu)$ remain transversal to the front $\Phi \subset \mathbb{R}^k$ for ν from the segment.*

Applying this lemma and Proposition 3.3 we obtain a sequence of equivariant homotopy equivalences

$$(F^-_N(\nu_* - \varepsilon), \partial F^-_N(\nu_* - \varepsilon)) \approx (F^-_N(\nu_* + \varepsilon), \partial F^-_N(\nu_* + \varepsilon))$$

and thus an isomorphism of semi-infinite cohomology $\mathcal{H}^*_{T^k}(F^-(\nu_* - \varepsilon)) \simeq \mathcal{H}^*_{T^k}(F^-(\nu_* + \varepsilon))$. The proof completes by choosing a finite subcovering of $[\nu_0, \nu_1]$ by such segments.

Our proof of the lemma is based on the fact that the front Φ of the family $\mathcal{A} = \{\mathcal{A}_\lambda\}$ of action functionals is $\mathbb{Z}^k$-periodic. The hyperplane $\Gamma(\nu_*) = p^{-1}(\nu_*)$ is transversal to Φ since ν_* is regular value of $\hat{p}$ (Proposition 3.1). Due to Corollary 3.2 we can choose $k-1$ coordinate hyperplanes in $\Gamma(\nu_*)$ such that they and all their intersections are transversal to $\Phi \cap \Gamma(\nu_*)$. Then all their integer translations are also transversal to Φ (periodicity!). Now we can choose an exhausting sequence of growing integer cubes framed by this coordinate net. Corresponding $\Gamma_N(\nu)$ and $\partial\Gamma_N(\nu)$ will be transversal to Φ for $\nu = \nu_*$ and thus for all ν in some neighborhood of ν_* (we use here openness of transversality provided that $\Phi/\mathbb{Z}^k$ is compact).

The same arguments apply even if ν_* is an isolated critical value of $\hat{p}$ but in this case $\Gamma(\nu_*)$ itself is tangent to Φ. In the proof of Proposition 4.2 one may assume without loss of generality that

(1) $\nu_0 = \nu_* - \varepsilon$, $\nu_1 = \nu_* + \varepsilon$,

(2) there is only one (lattice of) critical orbit(s) on the level ν_*,

(3) $(F_N^-(\nu_0), \partial F_N^-(\nu_0))$is embedded into$(F_N^-(\nu_1), \partial F_N^-(\nu_1))$ as the complement to a neighborhood of the critical orbit(s) (use isotopy of Proposition 3.3 outside of this neighborhood),

(4) the semi-infinite cohomology class $q \in \mathcal{R} \simeq \mathcal{H}^*_{T^k}(\mathbb{R}^{k-1}, \infty)$ is represented by a non-zero cohomology class

$$q \in H^*_{T^k}(F_N^-(\nu_1), \partial F_N^-(\nu_1))$$

which vanishes when restricted to $(F_N^-(\nu_0), \partial F_N^-(\nu_0))$.

The latter implies that q_1 is the image of some $\alpha \in H^*_{T^k}(F_N^-(\nu_1), F_N^-(\nu_0))$ under the connecting homomorphism δ in the long exact sequence

$$\begin{aligned} &\to H^{*-1}_{T^k}(F_N^-(\nu_1), F_N^-(\nu_0)) \xrightarrow{\delta} H^*_{T^k}(F_N^-(\nu_1), \partial F_N^-(\nu_1)) \to \\ &\to H^*_{T^k}(F_N^-(\nu_0), \partial F_N^-(\nu_0)) \to \dots \end{aligned}$$

The crucial point now is that in a neighborhood of the critical orbit the T^k-action is *free*—due to non-singularity of our initial toric manifold in fact. This implies that the equivariant cohomology $H^*_{T^k}(F_N^-(\nu_1), F_N^-(\nu_0))$ is isomorphic to the usual cohomology $H^*(F_N^-(\nu_1)/T^k, F_N^-(\nu_0)/T^k)$ of the quotient neighborhood of our critical orbit. The coefficient algebra $H^*_{T^k}(pt)$ acts trivially in the usual cohomology! Thus

$$v\alpha = 0 \text{ and } vc_1 = \delta(v\alpha) = 0.$$

Remark 6.4. In a similar manner one can show that in the long exact sequence of semi-infinite cohomology

$$\to \mathcal{H}^*_{T^k}(F^-(\nu_1), F^-(\nu_0)) \to \mathcal{H}^*_{T^k}(F^-(\nu_1)) \to \mathcal{H}^*_{T^k}(F^-(\nu_0)) \to$$

the left term is a free rank 1 $\mathbb{C}[\mathbb{Z}^{k-1}]$-module provided that the segment $[\nu_0, \nu_1]$ contains only one $\mathbb{Z}^{k-1}$-lattice of *non-degenerate* critical T^k-orbits.

Proof of Theorem 1.1. Our ship is fully loaded now.

Let $p : \mathbb{R}^k \to \mathbb{R}$ be a primitive integer value of the momentum map $\mathbb{C}^n \to \mathbb{R}^{k*}$. We are going to count critical levels of the function $\hat{p} : A \to \mathbb{R}$ between two non-singular levels ν and $\nu + 1$. On every critical level there is at least one $\mathbb{Z}^{k-1}$-lattice of critical T^k-orbits, all corresponding to the same fixed point of the hamiltonian transformation $-\operatorname{id} \circ h^1$ on our toric manifold M_p (Propositions 2.2, 4.3). To the same fixed point there correspond therefore ℓ critical levels of $\hat{p}$ between regular levels ν and $\nu + \ell$.

Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^k \subset \mathbb{Z}^n$ be positive, that is $m_j \geq 0$. We will show that the number $\#(m)$ of critical values of the function $\hat{p}$ between regular values ν and $\hat{\nu} = \nu + \langle p, m \rangle$ is not less than $\langle c, m \rangle$ $(= m_1 + \dots + m_n)$. This would imply that the total number of fixed points is not less than $\langle c, m \rangle / \langle p, m \rangle$.

Applying Corollary 1.3 to $\mathcal{J} = \mathcal{J}^*(F^-(\nu))$ (see (6.7)) we find some $q \in \mathcal{R}$ such that $q \notin \mathcal{J}$ but $u_1 q, \dots, u_n q \in \mathcal{J}$. If $\#(m) < m_1 + \dots + m_n$ then $u_1^{m_1} \dots u_n^{m_n} q \in \mathcal{J}^*(F^-(\hat{\nu}))$ (Propositions 6.2, 6.3).

On the other hand multiplication by u^m is the isomorphism (6.8): $\mathcal{J}^*(F^-(\nu)) \to \mathcal{J}^*(F^-(\hat{\nu}))$ and thus $u^m q \notin \mathcal{J}^*(F^-(\hat{\nu}))$.

This contradiction completes the proof.

Remark 6.5. One could prove the Morse-type estimate $\# \geq \dim H^*((M_p)$ for the number of non-degenerate fixed points of a hamiltonian transformation (without a reference to the Lefschetz theorem) combining Remark 6.4 with the fact that $\mathbb{C}[\mathbb{Z}^{k-1}]$-module

$$\mathcal{J}^*(F^-(\nu))/\mathcal{J}^*(F^-(\nu + 1))$$

has the same rank as $\mathcal{J}_0/\mathcal{J}_1$ (equal to $\dim H^*(M_p)$, see Remark 1.4).

References

[Ar1] V. I. Arnold, *Sur une propriété topologique des applications globalement canonique de la méchanique classique,* C. R. Acad. Sci. Paris 261 (1965), 3719–3722.

[Ar2] V. I. Arnold, *Mathematical methods of classical mechanics,* 2nd ed, Springer, N.Y. 1989.

[AVG] V. I. Arnold, A. N. Varchenko, S. M. Gusein-Zade, *Singularities of differentiable maps,* Birkhäuser, Boston, v.1 (1985), v.2 (1988).

[At] M. Atiyah, *Convexity and commuting hamiltonians,* Topology 23:1 (1984), 1–15.

[Au] M. Audin, *The topology of torus actions on symplectic manifolds,* Birkhäuser, Basel, 1991.

[Cha] M. Chaperon, *Une idée du type "geodésique brisées" pour les systèmes hamiltoniens,* C. R. Acad. Sci. Paris 298 (1984), 293–296.

[Che] Yu. V. Chekanov, *Legandrova teorija Morsa,* Uspekhi Mat. Nauk 42:4 (1987), 139–141.

[CZ] C. C. Conley, E. Zehnder, *The Birkhof-Lewis fixed point theorem and a conjecture of V. I. Arnold,* Invent. Math. 73:1 (1983), 33–49.

[F] B. L. Feigin, *Semi-infinite cohomology of Kac-Moody and Virsoro Lie algebras,* Russian Math. Surveys, 39:2 (1984), 155–156.

[Fl1] A. Floer, *Morse theory for Lagrangian intersections,* J. Diff. Geom. 28 (1988), 513–547.

[Fl2] A. Floer, *Symplectic fixed points and holomorphic spheres,* Commun. Math. Phys. 120 (1989), 575–611.

[FW] B. Fortune, A. Weinstein, *A symplectic fixed point theorem for complex projective spaces,* Bull. AMS 12:1 (1985), 128–130.

[G] V. A. Ginsburg, *Equivariant cohomology and Kahler geometry,* Funct. Anal. Appl. 21:4 (1987), 271–283.

[Gi1] A. B. Givental, *Periodic mappings in symplectic topology,* Funct. Anal. Appl. 23:4 (1989), 287–300.

[Gi2] A. B. Givental, *Nonlinear generalization of the Maslov*

index, Adv. in Sov. Math., v.1, AMS, Providence, RI, 1990, 71–103.
[GM] M. Goresky, R. MacPherson, *Stratified Morse theory,* Springer, N.Y., 1987.
[GrH] P. Griffits, J. Harris, *Principles of algebraic geometry,* Wiley, N.Y., 1978.
[Gro] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds,* Invent. Math. 82:2 (1985), 307–347.
[HoS] H. Hofer, D. A. Salamon, *Floer homology and Novikov rings,* Preprint, 1992, 39 p.
[Hs] W. Y. Hsiang, *Cohomology theory of topological transformation groups,* Springer, N.Y., 1975.
[KT] Y. Karshon, S. Tolman, *The moment map and line bundles over pre-symplectic toric manifolds,* Preprint, Harvard univ., 1992, 16 p.
[LS] F. Laudenbach, J.-C. Sikorav, *Présidence d´intersections avec la section nulle au course d´une isotopie hamiltonienne dans une fibré cotangent,* Invent. Math. 82:2 (1985), 349–358.
[Oh] Y.-G. Oh, *A symplectic fixed point theorem on* $T^{2n} \times \mathbb{C}P^k$, Math.-Z. 203:4 (1990), 535–552.
[V] A. A. Voronov, *Semi-infinite De Rham cohomology,* Preprint, MSRI, 1992, 29 p.

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