QUANTUM K-THEORY ON FLAG MANIFOLDS, FINITE-DIFFERENCE TODA LATTICES AND QUANTUM GROUPS

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0. Introduction

Let $X = \{0 \subset \mathbb{C}^1 \subset \ldots \subset \mathbb{C}^r \subset \mathbb{C}^{r+1}\}$ be the manifold of complete flags in $\mathbb{C}^{r+1}$. It admits the Plücker embedding into the product of projective spaces

$$X \hookrightarrow \Pi := \prod_{i=1}^{r} \mathbb{C}P^{n_i - 1}, \quad n_i := \binom{r+1}{i}.$$

Let $(x : y)$ be homogeneous coordinates on $\mathbb{C}P^1$. A degree d holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^N$ is uniquely determined, up to a constant scalar factor, by $N+1$ relatively prime degree d binary forms $(f_0(x : y) : \ldots : f_N(x : y))$. Omitting the condition that the forms are relatively prime we compactify the space of degree d holomorphic maps $\mathbb{C}P^1 \to \mathbb{C}P^N$ to a complex projective space of dimension $(N+1)(d+1)-1$. We denote this compactification of the space of maps by $\mathbb{C}P^N_d$. This construction defines the compactification $\Pi_d = \Pi_{i=1}^r \mathbb{C}P^{n_i-1}_{d_i}$ of the space of degree $d = (d_1, \ldots, d_r)$ maps from $\mathbb{C}P^1$ to Π.

Composing degree d holomorphic maps from $\mathbb{C}P^1$ to the flag manifold X with the Plücker embedding, we embed the space of such maps into Π_d. The closure QM_d of this space in Π_d is often referred to as the *Drinfeld's compactification* of the space of degree d maps from $\mathbb{C}P^1$ to X and will be called the space of *quasimaps* (following [14]). It is a (generally speaking — singular) irreducible projective variety of complex dimension $\dim X + 2d_1 + \ldots + 2d_r$.

The flag manifold is a homogeneous space of the group $SL_{r+1}(\mathbb{C})$ and of its maximal compact subgroup SU_{r+1}. The action of these groups on X lifts naturally to the spaces Π, Π_d, and QM_d. In addition to this action the spaces Π_d and the subspaces QM_d carry the circle action induced by the rotation of $\mathbb{C}P^1$ defined by $(x : y) \mapsto (x : e^{i\phi} y)$. Thus the product group $G = S^1 \times SU_{r+1}$ and its complex version $G_{\mathbb{C}} = \mathbb{C}^* \times SL_{r+1}(\mathbb{C})$ act on the quasimap spaces. We will see later that QM_d have G-equivariant desingularizations $\tilde{QM}_d$.

Date: August 15, 2001, Revised: August 10, 2002.

The research was partially supported by the NSF grants DMS-9704774 (A. G.) and DMS-0072547 (Y. L.) .

We denote by $P = (P_1, ..., P_r)$ the G-equivariant line bundles over QM_d obtained by pulling back the Hopf bundles over the complex projective factors of Π_d.

The G-equivariant holomorphic Euler characteristic of a G-equivariant holomorphic vector bundle V over a compact complex manifold M provided with a holomorphic G-action is defined as the character of the virtual representation of G in the alternating sum of the cohomology spaces:

$$\chi_G(V) := \mathrm{str}_G(H^*(M, V)) := \sum_k (-1)^k \,\mathrm{tr}_G(H^k(M, V)).$$

It can be expressed in cohomological terms using the equivariant version of the Riemann-Roch-Hirzebruch theorem:

$$Ch(\chi_G(V)) = \int_M Ch_G(V) Td_G(M),$$

where Ch_G and Td_G are the equivariant Chern character and Todd class, and Ch on the left hand side is the Chern character from $K_G^*(pt) = K^*(BG)$ (canonically isomorphic to a suitable completion of the character ring $Repr(G)$) to $H_G^*(pt) = H^*(BG)$.

We are interested in computing G-equivariant Euler characteristics of the line bundles $P^z = P_1^{z_1} ... P_r^{z_r}$ over equivariant desingularizations of the map spaces QM_d. In fact (see section 1) the result does not depend on the choice of desingularization. We encode the answers by the following generating function:

$$\mathcal{G}(Q, z, q, \Lambda) := \sum_d Q^d \chi_G(H^*(\tilde{QM}_d, P^z)) \tag{1}$$

Here q and $\Lambda = (\Lambda_0, ..., \Lambda_r | \Lambda_0 ... \Lambda_r = 1)$ are multiplicative coordinates on S^1 and on the maximal torus T^r of SU_{r+1} respectively, and $Q = (Q_1, ..., Q_r)$ are formal variables. The formula makes sense for integer values of $z = (z_1, ..., z_r)$ but can be extended to, say, complex values of z by interpreting the right hand side by means of the Riemann-Roch-Hirzebruch formula.

The *finite-difference Toda operator* [1]

$$\hat{H}_{Q,q} := q^{\partial/\partial t_0} + q^{\partial/\partial t_1}(1 - e^{t_0 - t_1}) + ... + q^{\partial/\partial t_r}(1 - e^{t_{r-1} - t_r}) \tag{2}$$

is composed from the operators of multiplication by $Q_i = e^{t_{i-1} - t_i}$ and from the translation operators

$$q^{\partial/\partial t_j} : t_i \mapsto t_i + \delta_{ij} \ln q.$$

We can now state the main result of the paper.

[1] We are thankful to P. Etingof for this definition.

Theorem 1. *In each of the two groups of variables Q and Q' the function*

$$G(Q,Q') := \mathcal{G}(Q, \frac{\ln Q' - \ln Q}{\ln q}, q, \Lambda)$$

is the eigen-function of the finite-difference Toda operator:

$$\hat{H}_{Q',q} G \ = \ (\Lambda_0 + ... + \Lambda_r) G \ = \ \hat{H}_{Q,q^{-1}} G \ .$$

Remarks. (1) Theorem 1 together with some, rather general, factorization property of the generating function $\mathcal{G}$ uniquely determines $\mathcal{G}$. We will prove this in Section 2 and also present there some explicit examples. The results of Section 2 link Theorem 1 with general concepts of quantum K-theory understood in the sense of [11] as K-theory on moduli spaces of stable maps. Combining the results of Section 2 with a reconstruction theorem from [22] applied to the flag manifolds we conclude that Theorem 1 determines — at least in principle — all genus 0 K-theoretic Gromov – Witten invariants and their gravitational descendents.

(2) Taking

$$q = \exp(-\hbar) = 1 - \hbar + \hbar^2/2 + ...$$

and assigning the degrees $\deg \hbar = 1, \deg Q_i = 2$ we perform the degree expansion of the finite-difference Toda operator:

$$\hat{H} = (r+1) - \hbar \sum \frac{\partial}{\partial t_i} + [\frac{\hbar^2}{2} \sum \frac{\partial^2}{\partial t_i^2} - \sum e^{t_{i-1} - t_i}] + ...$$

In degrees 1 and 2 the expansion spits out the *momentum* and the *Hamilton* operators of the quantum Toda lattice thus explaining the name of $\hat{H}$. In fact the Toda operator $\hat{H}$ can be included into a complete set of commuting finite-difference operators ("conservation laws") in close analogy with the case of quantum and classical Toda systems. A construction of such operators in terms of quantum groups is described in Section 5.

(3) As we will see in Section 2, the generating function G is a common eigenfunction of the commuting conservation laws of the finite-difference Toda system. This result is a K-theoretic counterpart (in the case of the series A_r) of the theorem by B. Kim [17] characterizing intersection theory in spaces of holomorphic maps $\mathbb{C}P^1 \to G/B$ in terms of quantum Toda lattices (see Section 5.1 for more details).

(4) A conjecture generalizing Theorem 1 to the case of flag manifolds $X = G/B$ of arbitrary semi-simple complex Lie groups G and intertwining K-theory on moduli spaces of stable maps with representation theory of quantum groups is explained in Section 5 (which can read directly after Sections 1 and 2).

(5) The results of the paper were completed in the Summer 98 and reported by the authors at a number of conferences and seminars. In this version of the paper we decided to leave the material of Section 5 in the form close to the preliminary text written in 1998. In particular, we did not try to match the quantum group

description of finite-difference Toda lattices given in Section 5 with the (apparently very similar) construction that has become standard since then due to the paper [7] by P. Etingof. Perhaps partly motivated by our conjectures, the paper places the finite-difference theory of Whittaker functions on foundations much more solid than those available to us in 1998.

(6) Initially the conjecture proved in this paper served as a motivation for developing basics [11, 21] of *quantum K-theory* — a K-theoretic counterpart of quantum cohomology theory. Moreover, one can heuristically interpret the generating function $\mathcal{G}$ as an object of Floer-type (or semi-infinite) K-theory on the loop space LX equipped with the S^1-action defined by the rotation of loops. According to [9] this heuristics, applied in cohomology theory, suggests existence of a D-module structure (which in the case of flag manifolds is identified by Kim's theorem [17] with the quantum Toda system). The same arguments in K-theory lead to a finite-difference counterpart of the D-module structure. While results of the present paper conform with this philosophy, the role of finite-difference equations in the general structure of quantum K-theory remains uncertain (see Section 5 (d) in [10] for a few more details on this issue).

(7) The proof of Theorem 1 is presented in Sections 1 – 4. In Section 1 we discuss independence of the function $\mathcal{G}$ on desingularizations. In Section 2 we use desingularizations of the quasimap spaces based on moduli spaces of stable maps in order to obtain the factorization and other properties of the function $\mathcal{G}$ mentioned in the Remarks 1, 3. In Section 3 we describe the *hyperquot schemes* — another equivariant desingularizations of the spaces QM_d — and compute their equivariant canonical class. This result plays a key role in our derivation of Theorem 1 given in Section 4.

1. Rational desingularizations

A germ (M, p) of a complex irreducible algebraic variety is called *rational* if for any desingularization $\pi : (M', p') \to (M, p)$ all higher direct images of the structure sheaf of M' vanish in a neighborphood of p:

$$(R^k \pi_* \mathcal{O}_{M'})_p = 0 \text{ for all } k > 0, \tag{3}$$

and $R^0 \pi_* \mathcal{O}_{M'} = \mathcal{O}_M$.

A non-singular germ is rational. This is a rephrasing of the famous Grothendieck conjecture [13] proved by Hironaka [16] on the basis of his resolution of singularity theorem and saying that (3) holds true for any $f : M' \to M$ which is a proper birational isomorphism of non-singular spaces.

Another easy consequence [15] of the Hironaka theorem is that the condition (3) in the definition of rational singularities is automatically satisfied for all desingularizations if it is satisfied for one of them. [2] In particular, it makes obvious the fact that the product of a rational singularity with a non-singular space has rational singularities.

Let us call a *rational desingularization* of a singular space N a proper birational isomorphism $M \to N$ where M is allowed to have rational singularities at the worst. Consider a compact irreducible complex variety N and two birational desingularization $g_i : M_i \to N,\ i = 1, 2$. For any vector bundle V on N we have $\chi(M_1; g_1^* V) = \chi(M_2, g_2^* V)$. Indeed, if $f_i : M \to M_i$ is a common desingularization (so that $f_1 \circ g_1 = f_2 \circ g_2$) then

$$\chi(M, f_i^* g_i^* V) = \chi(M_i, (R^* f_i)_* \mathcal{O}_M \otimes g_i^* V) = \chi(M_i, \mathcal{O}_{M_i} \otimes g_i^* V) = \chi(M_i, g_i^* V).$$

In our applications, we take on the role of N the quasimap spaces QM_d of the space of maps $\mathbb{C}P^1 \to X$. The holomorphic Euler characteristics we are really interested in are those for bundles on QM_d pulled back to the graph spaces (see Section 2). For flag manifolds X the graph spaces are orbifolds, and it is important for us that, due to a theorem by Viehweg [27], *singularities of orbifolds are rational.*

We need however the following equivariant refinement of the above independence argument:

Proposition 1. *Let a compact connected Lie group G act by holomorphic transformations on a compact irreducible complex projective variety N and on two of its equivariant projective rational desingularizations $f_i : M_i \to N,\ i = 1, 2$. Then for any G-equivariant holomorphic vector bundle V on N we have $\chi_G(M_1, f_1^* V) = \chi_G(M_2, f_2^* V)$.*

Proof. Proposition 1 would immediately follow from an equivariant version of the Hironaka resolution of singularities theorem. Since we do not know a suitable reference, we derive Proposition 1 from its non-equivariant version as follows.

The equivariant holomorphic Euler characteristic is a character of a virtual representation of G and is determined by its restriction to the maximal torus in G. Therefore we may assume that G is such a torus. Furthermore, the characters are determined by their (sufficiently high order) jets at the point $1 \in G$. These jets have the following interpretation in equivariant K-theory.

Let $BG^{(n)} := (\mathbb{C}P^n)^s$ be the finite-dimensional approximations to the classifying space $BG = (\mathbb{C}P^\infty)^s$ of the s-dimensional torus G. Let $\pi_i : (M_i)_G^{(n)} \to BG^{(n)}$ be M_i-bundles associated with the restriction of the universal G-bundle to $BG^{(n)}$. Similarly, consider the associated N-bundle $\pi : N_G^{(n)} \to BG^{(n)}$, the bundle maps $F_i : (M_i)_G^{(n)} \to N_G^{(n)}$ induced by the equivariant maps $f_i : M_i \to N$ and the vector

[2] This is proved by applying Leray spectral sequences to the commutative square formed by three desingularizations $M' \to M$, $M'' \to M$ and $M''' \to M$ dominating the first two: $M''' \to M', M''' \to M''$. We are thankful to R. Hartshorne for teaching us this subject.

bundle $V^{(n)}$ over $N_G^{(n)}$ associated with the equivariant bundle V over N. Then the K-theoretic push-forwards $(\pi_i)_*(F_i^* V^{(n)})$ (which are elements in the Grothendieck group $K^*(BG^{(n)}$ defined as the alternated sums $\sum(-1)^k \sum R^k(\pi_i)_*(...)$ of higher direct images) coincide with suitable jets of the characters $\chi_G(M_i, f_i^* V)$. [3] Moreover, these elements are uniquely determined by their K-theoretic Poincare pairing with elements of $K^*(BG^{(n)})$, i. e. by holomorphic Euler characteristics of the form

$$\chi((M_i)_G^{(n)}, F_i^*(V^{(n)} \otimes \pi^* W)) ,$$

where the vector bundles W run a basis in $K^*(BG^{(n)})$.

The bundles π_i are locally trivial with fibers M_i having only rational singularities. Therefore their total spaces $(M_i)_G^{(n)}$ have only rational singularities too. The proposition follows now from its non-equivariant version applied to the rational desingularizations F_i of the spaces $N_G^{(n)}$.

2. Graph spaces and factorization

2.1. Let X be the complete flag manifold as in the Introduction. The graph of a degree d holomorphic map $\mathbb{C}P^1 \to X$ is a genus 0 compact holomorphic curve in $GX := \mathbb{C}P^1 \times X$ of *bi-degree* $(1, d)$, i. e. of degree 1 in projection to $\mathbb{C}P^1$ and of degree d in projection to X. We define the *graph space* GX_d as the moduli space of genus 0, unmarked stable maps to GX of bi-degree $(1, d)$. For $X = G/B$, the graph spaces GX_d, according to [18, 4, 8], are compact complex projective [4] algebraic orbifolds of dimension $2d_1 + ... + 2d_r + \dim X$. They provide therefore compactification of spaces of degree d holomorphic maps $\mathbb{C}P^1 \to X$ and inherit the action of $G_{\mathbb{C}} = S^1_{\mathbb{C}} \times SL_{r+1}(\mathbb{C})$ from the componentwise action on target space.

The natural birational isomorphism between the graph spaces and quasimap spaces is actually defined by a regular map (see for example [10]) which can be described as follows. A bi-degree $(1, d)$ rational curve in $\mathbb{C}P^1 \times X$ projected to $\mathbb{C}P^1 \times \mathbb{C}P^{n_i - 1}$ by the Plücker map consists of the graph Σ_0 of a degree $m_0 \leq d_i$ map $\mathbb{C}P^1 \to \mathbb{C}P^{n_i-1}$ and a few "vertical" curves Σ_j of bi-degrees $(0, m_j)$ with $\sum m_j = d_i - m_0$, attached to the graph. The graph component is given by n_i mutually prime binary forms of degree m_0. Denote $\zeta_j \in \mathbb{C}P^1$ the images of the vertical curves Σ_j in projection to $\mathbb{C}P^1$. Multiplying the binary forms by the

[3] The coincidence, tautological in topological equivariant K-theory [2], holds true in the context of algebraic geometry for non-singular complex manifolds due to the compatibility of algebraic-geometrical and topological K-theoretic push-forwards (see [3]). We need here a more general statement applicable to possibly singular spaces M. We don't have a suitable general reference and assume that M are projective instead. Then one can use an equivariant embedding of M into a projective space P in order to push-forward f^*V from $K_G^0(M)$ to $K_G^0(P)$ and then apply the coincidence in question for the non-singular space P.

[4] See, for instance, [8] where projectivity of moduli spaces of stable maps to complex projective manifolds is proved. We need this property only to assure that Proposition 1 applies to the graph spaces considered as equivariant rational desingularizations of the quasimap spaces.

common factor with roots of multiplicity m_j at ζ_j we obtain the degree d_i vector-valued binary form which specifies the image of our curve in $\mathbb{C}P^{(d_i+1)n_i-1}$. The regular map

$$\mu: \; GX_d \to \Pi_d$$

is defined by the above construction applied to each component of the Plücker embedding.

2.2. The following result allows to separate the variables Q and Q' in the generating function $G(Q,Q')$. Let $p_1, ..., p_r$ denote Hopf line bundles on Π pulled back to the flag manifold X by the Plücker embedding, and $p^z = p_1^{z_1}...p_r^{z_r}$ be their tensor product. Denote by $\langle V, W\rangle := \chi_G(X; V \otimes W)$ the K-theoretic Poincaré pairing on $K_G^*(X)$.

Proposition 2.

$$G(Q,Q') = \langle J(Q',q)p^{\ln Q'/\ln q}, p^{-\ln Q/\ln q} J(Q,q^{-1})\rangle,$$

where J is a suitable formal Q-series with coefficients in $K_G^*(X) \otimes \mathbb{Q}(\Lambda, q)$ (described below) .

In the description of the series J, and in the proof of Proposition 2 we will encounter the following concepts standard in Gromov – Witten theory (see for instance [18, 4] for their definitions and properties). We will use the symbol $X_{m,d}$ for the moduli space of genus 0 degree d stable maps $f : (\Sigma, \varepsilon_1, ...\varepsilon_m) \to X$ with m marked points (and we intend to avoid the notation $(\mathbb{C}P^1 \times X)_{0,(1,d)}$ for the graph spaces GX_d). Let L denote the *universal cotangent line* bundle over $X_{1,d}$ formed by the cotangent lines $T_\varepsilon^*\Sigma$ to the curves (Σ, ε) at the marked point ε (see [11] for a discussion of these line bundles in the context of K-theory). Let $\mathrm{ev}_* : K_G^*(X_{1,d}) \to K_G^*(X)$ be the K-theoretic push-forvard by the map $\mathrm{ev} : X_{1,d} \to X$ defined by the evaluation $f \mapsto f(\varepsilon)$ at the marked point. In these notations

$$J(Q,q) = 1 + \frac{1}{1-q} \sum_{d \neq 0} Q^d \, \mathrm{ev}_*(\frac{1}{1-Lq}).$$

Proof. It is based on localization to fixed points of the S^1-action on the graph spaces GX_d and goes through with minor modifications in the general setting of quantum K-theory described in [11].

Let us begin with some general remarks on K-theoretic fixed point localization for S^1-actions on *orbifolds* (the generalization to tori actions is immediate but is not needed here). When V is an S^1-equivariant complex bundle on a closed complex *manifold* M, we have the following Bott – Lefschetz localization formula

$$\chi_{S^1}(M;V) = \chi_{S^1}(M^{S^1}; \frac{i^*V}{Euler_{S^1}(N)}) = \chi(M^{S_1}; \frac{\sum q^n (i^*V)^{(n)}}{\prod_n Euler(q^n N^{(n)})}).$$

Here N is the normal bundle to the fixed point submanifold M^{S^1}, $i : M^{S^1} \to M$ is the embedding, $i^*V = \sum (i^*V)^{(n)}$ and $N = \sum N^{(n)}$ are decompositions by the characters $q^n,\ n \in Z$, of the S^1-action, and $Euler(W) := 1 - W^* + \wedge^2 W^* - ...$ is the "Koszul complex" of a bundle W. The restriction homomorphism i^* is known to be an isomorphism over the field of fractions of the coefficient ring $K^*_{S^1}(pt)$. The localization formula follows therefore from the identity $i^* i_* 1 = Euler_{S^1}(N)$.

In the orbifold / orbibundle situation the above argument goes through. However the canonical lift of the infinitesimal action of S^1 on M and V to local non-singular charts, when integrated to a circle action, may lead to a "larger circle". As a result, the orbibundles i^*V and N on M^{S^1} decompose into the characters $q^{n/m},\ n \in Z$, of certain m-fold cover of S^1. By the definition of holomorphic Euler characteristic on orbifolds $\chi_{S^1}(M^{S^1}, W)$ is the $\mathbb{Z}_m$-invariant part of $\sum_{n\in\mathbb{Z}} q^{n/m} \chi(M^{S^1}; W^{(n/m)})$. Thus only the integer powers of q are to be confined on the RHS of the localization formula or, equivalently, *the average over all the m values of $q^{1/m}$ should be taken.*

Having stated the genaral rule we should point out that Proposition 2 deals only with curves of degree 1 in projection to $\mathbb{C}P^1$, which implies $m = 1$ so that the aforementioned modification of the fixed point localization formula does not occur.

Let $f : \Sigma \to \mathbb{C}P^1 \times X$ be a bi-degree $(1, d)$ stable map which represents in GX_d a fixed point of the S^1-action. Then f consists of the graph of a constant map $\mathbb{C}P^1 \to X$ and of two stable maps $f_\pm : (\Sigma_\pm, \varepsilon_\pm) \to X_\pm$ of bi-degrees $(0, d^\pm)$, $d^+ + d^- = d$, with one marked point $\varepsilon_\pm$ each, attached to the graph at the points $f_\pm(\varepsilon_\pm)$. Here $X_\pm$ are two copies of X, namely the slices of the product $\mathbb{C}P^1 \times X$ over the fixed points $(1 : 0)$ and $(0 : 1)$ of the S^1-action on $\mathbb{C}P^1$. In the extremal cases $d^+ = 0$ or (and) $d^- = 0$ only one (none) of the maps $f_\pm$ is present. A fixed point component in GX_d is therefore identified with the suborbifold in the product $X_{1,d^+} \times X_{1,d^-}$ of moduli spaces of stable maps to X with one marked point, given by the diagonal constraint $\mathrm{ev}(f_+) = \mathrm{ev}(f_-)$ in $X \times X$ for evaluations at the marked points.

According to the construction of the map $\mu : GX_d \to \Pi_d$, the image $\mu[f]$ is represented by an r-tuple of *monomial* binary vector-forms with zeroes of orders d^+ and d^- at $(1 : 0)$ and $(0 : 1)$ respectively.

The S^1-fixed points in Π_d represented by these vector-forms form a fixed point submanifolds isomorphic to Π. We denote it by $\Pi_d^{(d_+)}$ as the fixed point set $\Pi_d^{S^1}$ consists of one copy of Π for each $0 \le d_+ \le d$. The images $\mu[f]$ of the above fixed from GX_d form a copy of X Plücker-embedded into $\Pi_d^{(d_+)}$. [5] This implies that the S^1-equivariant line bundles $P = (P_1, ..., P_r)$ restricted to the fixed point component coincide with $\mathrm{ev}^*(p) \otimes q^{d^+}$. Here $p = (p_1, ..., p_r)$ is the list of Hopf line bundles on X, and $q^m = (q^{m_1}, ..., q^{m_r})$ specify the S^1-actions on the r-tuple of trivial line bundles.

[5] We will need this information and notations also in Section 4.

Each of the vertical curves $f_\pm$ contributes a 2-dimensional summand into the conormal bundle to the fixed point component. The two infinitesimal deformations of $[f_\pm]$ breaking the S^1-invariance correspond to the shift of the vertical curve away from the slice $X_\pm$ and to the smoothening of Σ at the nodal point $\varepsilon_\pm$. These deformations contribute respectively the factors $(1-q^{\pm 1})$ and $(1-L_\pm \otimes q^{\pm 1})$ to the denominator of the Bott–Lefschetz localization formula. Here $L_\pm$ is the universal cotangent line bundle over $X_{1,d^\pm}$ formed by cotangent lines $T^*_{\varepsilon_\pm}\Sigma_\pm$. The contribution of the fixed point component to the localization formula can be therefore written as

$$\langle (\mathrm{ev}_+)_*[(1-q)^{-1}(1-L_+q)^{-1}]\ p^z q^{d^+z},\ (\mathrm{ev}_-)_*[(1-q^{-1})^{-1}(1-L_-q^{-1})^{-1}]\rangle,$$

(where however the factor $(\mathrm{ev}_\pm)_*[...]$ should be omitted if $d^\pm = 0$). We use here transversality of $\mathrm{ev}_+ \times \mathrm{ev}_-$ to the diagonal $i : \Delta \subset X \times X$, which allows us to replace the push-forward to $X_{1,d^+} \times X_{1,d^-}$ of the structure sheaf of the fixed point component by $(\mathrm{ev}_+ \times \mathrm{ev}_-)^*(i_*(O_\Delta))$.

Summing the contributions over all (d^+, d^-) with the weights $Q^{d^+ + d^-}$ we find

$$\mathcal{G} = \langle J(Qq^z, q), p^z J(Q, q^{-1})\rangle.$$

It remains to recall that $G(Q, Q')$ is transformed to $\mathcal{G}$ by the substitution $Q' = Qq^z$.

2.3. Proposition 2 shows that Theorem 1 has the following reformulation.

Theorem 2. *The $K^*_G(X)$-valued vector-series $p^{\ln Q/\ln q} J(Q, q)$ is the eigen-vector of the finite-difference Toda operator $\hat{H}_{Q,q}$ with the eigen-value $\Lambda_0^{-1} + ... + \Lambda_r^{-1}$.*

The series J turns into 1 when reduced modulo Q. In particular, for the manifold X of complete flags in $\mathbb{C}^{r+1}$ application of the operator $\hat{H}_{Q,q}$ to $p^{\ln Q/\ln q} J$ yields, modulo Q, the factor $p_1 + p_1^{-1}p_2 + ... + p_{r-1}^{-1}p_r + p_r^{-1}$. The factor represents the trivial bundle $\mathbb{C}^{r+1}$ and is thus equal to $\Lambda_0^{-1} + ... + \Lambda_r^{-1}$ in $K^*_G(X)$.

Forgetting the space $\mathbb{C}^i$, $i = 1, ..., r$, in the flag $\mathbb{C}^1 \subset ... \subset \mathbb{C}^r \subset \mathbb{C}^{r+1}$ defines r projections $X \to X^{(i)}$ to partial flag manifolds with the fiber $\mathbb{C}P^1$. The degrees $\mathbf{1}_1, ..., \mathbf{1}_r$ of the fibers considered as rational curves in X form the basis in $H_2(X)$ dual to the basis $(-c_1(p_1), ..., -c_1(p_r))$ in $H^2(X)$ (and are represented by the monomials $Q_1, ..., Q_r$ in our generating series). This identifies $X^{(i)}$ with the moduli space $X_{0,\mathbf{1}_i}$, the projection $X \to X^{(i)}$ — with the forgetting map $\mathrm{ft} : X_{1,\mathbf{1}_i} \to X_{0,\mathbf{1}_i}$, and shows that the evaluation map $\mathrm{ev} : X_{1,\mathbf{1}_i} \to X$ is an isomorphism. This information about curves of minimal degrees in X allows us to compute J modulo $(Q)^2$:

$$J = 1 + \frac{1}{1-q}\Big[\frac{Q_1}{1-p_1^2p_2^{-1}q} + ... + \frac{Q_i}{1-p_{i-1}^{-1}p_i^2p_{i+1}^{-1}q} + ... + \frac{Q_r}{1-p_{r-1}^{-1}p_r^2 q}\Big]\ +\ o(Q).$$

The finite-difference equation $\hat{H}I = (\sum \Lambda_j^{-1})I$ for $I = p^{\ln Q/\ln q}J$ is equivalent to the recursion relation

$$[p_1(q^{d_1}-1) + ... + p_ip_{i-1}^{-1}(q^{d_i-d_{i-1}}-1) + ... + p_r^{-1}(q^{-d_r}-1)]J_d =$$

$$p_2 p_1^{-1} q^{d_2-d_1} J_{d-\mathbf{1}_1} + ... + p_r^{-1} q^{-d_r} J_{d-\mathbf{1}_r} \tag{4}$$

for the coefficients $J_d = (1-q)^{-1}\,\mathrm{ev}_*(1-qL)^{-1}$ of the series $J = \sum J_d Q^d$. It is straightforward now to verify the relation for $d = \mathbf{1}_i$.

The relation (4) recursively determines the coefficients J_d unambiguously, which implies

Corollary 1. *The power Q-series J is uniquely determined by Theorem 2 and by the constant term $J|_{Q=0} = 1$.*

Moreover, the uniqueness argument applies to arbitrary eigen-functions of $\hat{H}$. Let $\hat{D}$ be any finite-difference operator with coefficients polynomial in Q which commutes with $\hat{H}$, and let $I = p^{\ln Q/\ln q} \sum_{d\geq 0} I_d Q^d$ be a power Q-series with vector-coefficients $I_d \in K_G^*(X)$. The following statement is the finite-difference version of Kim's lemma [17] important in quantum cohomology theory of flag manifolds.

Lemma. *If I is an eigen-function of $\hat{H}$: $\hat{H}I = \Lambda_H I$, and $\hat{D}I$ is proportional to I modulo Q: $\hat{D}I \equiv \Lambda_D I \mod Q$, then I is an eigen-function of $\hat{D}$: $\hat{D}I = \Lambda_D I$.*

Indeed, $\hat{H}\hat{D}I = \hat{D}\hat{H}I = \Lambda_H(\hat{D}I)$ and therefore $\hat{D}I$ is an eigen-function of $\hat{H}$ with the same constant term $\Lambda_D I_0$ as $\Lambda_D I$ and thus coincides with it due to the uniqueness argument.

The conservation laws of finite-difference Toda systems discussed in Section 5 satisfy the hypotheses of the lemma, and we obtain

Corollary 2. *The vector-function I (as well as the series G) is a common eigen-function of the commuting conservation laws of the finite-difference Toda system.*

2.4. Example: $r = 1$. The problem of computing the series (1) can be generalized to arbitrary compact Kähler manifold X. In the case $X = \mathbb{C}P^r$ the quasimap spaces coincide with the projective spaces $\mathbb{C}P^r_d$ and are non-singular. The corresponding series (1) can be computed immediately by the holomorphic Bott – Lefschetz formula. We have $\sum_{d=0}^{\infty} Q^d \chi_G(\mathbb{C}P^r_d; P^{\otimes z}) =$

$$= -\sum_{d=0}^{\infty} \frac{Q^d}{2\pi i} \oint_{P\neq 0} \frac{P^{z-1}\, dP}{\prod_{j=0}^r \prod_{m=0}^d (1 - P\Lambda_j q^{-m})} = \langle J(Qq^z, q), p^z J(Q, q^{-1})\rangle,$$

where

$$\langle \Phi, \Psi\rangle = -\frac{1}{2\pi i}\oint_{p\neq 0} \frac{\Phi(p)\Psi(p)dp}{p\Pi_{j=0}^r (1-p\Lambda_j)}, \quad \text{and} \quad J = \sum_{d=0}^{\infty} \frac{Q^d}{\prod_{j=0}^r \prod_{m=1}^d (1-p\Lambda_j q^m)}.$$

The contour of integration here includes all poles except 0. The Hopf line bundle p satisfies the relation $(1-p\Lambda_0)...(1-p\Lambda_r) = 0$ in $K_G^*(\mathbb{C}P^r)$.

In the non-equivariant limit $\Lambda_0 = ... = \Lambda_r = 1$ the vector-function $I := p^{\ln Q/\ln q} J(Q, q)$ satisfies the finite-difference equation $D^{r+1} I = QI$ (here $D\, I(Q) :=$

$I(Q)-I(qQ))$ with the symbol resembling the famous relation in the quantum cohomology algebra of $\mathbb{C}P^r$.

In the special case of the manifold $\mathbb{C}P^1$ of complete flags in $\mathbb{C}^2$ the series $I := p^{\ln Q/\ln q} J$ at $Q=\exp(t_0-t_1)$ reads

$$I = p^{\frac{t_0-t_1}{\ln q}} \sum_{d=0}^{\infty} \frac{e^{d(t_0-t_1)}}{\prod_{m=1}^{d} (1-p\Lambda_0 q^m)(1-p\Lambda_0^{-1}q^m)}$$

and modulo $(1-p\Lambda_0)(1-p\Lambda_0^{-1})=0$ satisfies

$$[q^{\partial/\partial t_0} + q^{\partial/\partial t_1}(1-e^{t_0-t_1})] \ \ I = (\Lambda_0+\Lambda_0^{-1}) \ I.$$

2.5. Example: $r=2$. The space of complete flags in $\mathbb{C}^3$ coincides with the incidence relation (line) $\subset$ (hyperplane) in $\mathbb{C}P^2 \times \mathbb{C}P^{2*}$. In this case the series $I := p^{\ln Q/\ln q} J$ still can be written explicitly in terms of the algebra $K_G^*(\mathbb{C}P^2 \times \mathbb{C}P^{2*})$ described by the relations

$$(1-p_1\Lambda_0)(1-p_1\Lambda_1)(1-p_1\Lambda_2)=0,\ (1-\frac{p_2}{L_0})(1-\frac{p_2}{L_1})(1-\frac{p_2}{L_2})=0,\ \Lambda_0\Lambda_1\Lambda_2=1.$$

We have

$$I = p_1^{\frac{t_0-t_1}{\ln q}} p_2^{\frac{t_1-t_2}{\ln q}} \sum_{d_1,d_2=0}^{\infty} \frac{e^{d_1(t_0-t_1)+d_2(t_1-t_2)} \ \prod_{m=0}^{d_1+d_2}(1-p_1p_2q^m)}{\prod_{j=0}^{2}[\prod_{m=1}^{d_1}(1-p_1\Lambda_j q^m)\prod_{m=1}^{d_2}(1-p_2\Lambda_j^{-1}q^m)]}.$$

It is not hard to check directly that modulo the relations the series satisfies

$\hat{H}I = (\Lambda_0^{-1}+\Lambda_1^{-1}+\Lambda_2^{-1})I$, where $\hat{H} = q^{\partial/\partial t_0} + q^{\partial/\partial t_1}(1-e^{t_0-t_1}) + q^{\partial/\partial t_2}(1-e^{t_1-t_2})$,

and that $\mathcal{G} = \langle I(Qq^z,q), I(Q,q^{-1})\rangle =$

(5)
$$\sum_{d_1,d_2=0}^{\infty} \frac{Q_1^{d_1}Q_2^{d_2}}{(2\pi i)^2} \oint_{P_1\neq 0}\oint_{P_2\neq 0} \frac{P_1^{z_1-1}P_2^{z_2-1} \quad \prod_{m=0}^{d_1+d_2}(1-P_1P_2q^{-m}) \ \ dP_1\wedge dP_2}{\prod_{j=0}^{2}[\prod_{m=0}^{d_1}(1-P_1\Lambda_j q^{-m}) \ \prod_{m=0}^{d_2}(1-P_2\Lambda_j^{-1}q^{-m})]}.$$

3. Hyperquot schemes and the canonical class

3.1. A degree d holomorphic map $\mathbb{C}P^1 \to X$ defines a flag of subbundles $E^1 \subset ... \subset E^r \subset \mathbb{C}^{r+1}$ in the trivial bundle over $\mathbb{C}P^1$ of degrees $c_1(E^1) = -d_1, ..., c_1(E^r) = -d_r$. The *hyperquot scheme* HQ_d is defined as the moduli space of the diagrams $E^1 \to ... \to E^r \to E^{r+1} = \mathbb{C}^{r+1}$ where the morphisms $E^i \to E^{i+1}$ of vector bundles are injective almost everywhere on $\mathbb{C}P^1$. According to [5, 6] the hyperquot schemes are compact non-singular algebraic manifolds.

More generally, the construction applies to partial flag manifolds and to grassmannians (in which case one obtains Grothendieck's quot-schemes studied in [26]) and thus provides some non-singular compactifications of spaces of parameterized rational holomorphic curves in these spaces. In the case of degree $-d$ line subbundles in $\mathbb{C}^N$ the quot-scheme coincides with the projectivization $\mathbb{C}P_d^N$ of the space of vector-valued binary forms described in the Introduction.

Replacing the flag $E^1 \to ... \to E^r \to \mathbb{C}^{r+1}$ by the top exterior powers $\wedge^i E^i \to \wedge^i \mathbb{C}^{r+1}$ we obtain a natural map $\lambda : HQ_d \to \Pi_d := \Pi_{i=1}^r \mathbb{C}P_{d_i}^{n_i-1}$ to the product of the quot-schemes. The image of this map is the quasimap space QM_d. According to [5, 6], the hyperquot schemes are non-singular compact algebraic manifolds. Therefore the map

$$\lambda : HQ_d \to QM_d \subset \Pi_d$$

provides an equivariant desingularization of the quasimap space. [6]

3.2. Denote K_d the canonical line bundle of the hyperquot scheme HQ_d. The following description of K_d in terms of the pull-backs $P_1, ..., P_r$ of the Hopf line bundles from Π_d will play a key role in the proof of Theorem 1. Recall that q denotes the generator in the coefficient ring $K_{S^1}^*(pt)$ of the S^1-equivariant K-theory on HQ_d.

Theorem 3. *The class of the canonical line bundle K_d in $K_G^*(HQ_d) \otimes \mathbb{Q}$ coincides with the pull-back by λ of*

$$q^{-k_d} P_1^{2+2d_1-d_2} P_2^{2-d_1+2d_2-d_3} ... \, P_{r-1}^{2-d_{r-2}+2d_{r-1}-d_r} P_r^{2-d_{r-1}+2d_r},$$

where

$$k_d = d_1 + ... + d_r + \sum \frac{(d_i - d_{i-1})^2}{2}.$$

Proof. The moduli spaces HQ_d is naturally equipped with the *universal flag* (see [5, 6]) $\mathcal{E}^1 \subset ... \subset \mathcal{E}^{r+1} = \mathcal{O}^{r+1}$ of locally free sheaves on the product $\mathbb{C}P^1 \times HQ_d$ (such that restrictions to $\mathbb{C}P^1 \times \{b\}$ are sheaves of sections of the bundles in the diagram $E^1 \to ... \to E^{r+1} = \mathbb{C}^{r+1}$ representing $b \in HQ_d$). According to [5, 6], the tangent sheaf $\mathcal{T}_d$ of the hyperquot scheme can be described as the kernel of the following surjection:

$$\sum_i f_{i-1}^* \otimes \mathrm{Id} - \mathrm{Id} \otimes g_i : \oplus_i Hom(\mathcal{E}^i, \mathcal{O}^{r+1}/\mathcal{E}^i) \to \oplus_i Hom(\mathcal{E}^i, \mathcal{O}^{r+1}/\mathcal{E}^{i+1}).$$

Here $f_i : \mathcal{E}^i \to \mathcal{E}^{i+1}$ are the inclusions and $g_i : \mathcal{O}^{r+1}/\mathcal{E}^i \to \mathcal{O}^{r+1}/\mathcal{E}^{i+1}$ are corresponding projections. In other words, the class of the tangent bundle $\mathcal{T}_d$ to HQ_d in the Grothendieck group $K_{S^1 \times G}^*(HQ_d)$ equals

$$\oplus_{i=1}^r [\ Ext^0(\mathbb{C}P^1; \mathcal{E}^i, \mathcal{E}^{r+1}/\mathcal{E}^i) \ominus Ext^0(\mathbb{C}P^1; \mathcal{E}^i, \mathcal{E}^{r+1}/\mathcal{E}^{i+1})\ ].$$

It is easy to see from long exact sequences generated by $0 \to \mathcal{E}^j \to \mathcal{E}^{r+1} \to \mathcal{E}^{r+1}/\mathcal{E}^j \to 0$ that $Ext^1(\mathbb{C}P^1; \mathcal{E}^j, \mathcal{E}^{r+1}/\mathcal{E}^j) = 0$. Thus the class of tangent bundle is represented by the K-theoretic push-forward along the projection $pr : \mathbb{C}P^1 \times HQ_d \to HQ_d$:

$$\mathcal{T}_d = pr_*[\ \oplus_i (\mathcal{E}^i)^* \otimes (\mathcal{E}^{i+1} - \mathcal{E}^i)\ ].$$

[6] In fact [20] it is a small resolution.

We intend to compute the equivariant 1-st Chern class of $\mathcal{T}_d$ by means of the relative Riemann – Roch theorem:

$$Ch(pr_* V) = pr_*[Ch(V)Td(\mathbb{C}P^1)],$$

where Ch and Td are equivariant Chern character and Todd class respectively. Since fibers of pr have dimension 1, it suffices to confine only terms of degree ≤ 2 in $Ch(V)$ and $Td(\mathbb{C}P^1)$ in order to compute the 1-st Chern class of $pr_*(V)$. Our computation goes through due to the following "miracle": for a virtual bundle V of the form $V = \sum (E^i)^*(E^{i+1} - E^i)$, where E^i have dimensions i and E^{r+1} is trivial, the degree ≤ 2 terms of $Ch(V)$ depend only on 1-st Chern classes $c_1(E^i)$. Namely,

$$Ch(V) = \frac{r(r+1)}{2} - 2\sum c_1(E^i) + \frac{1}{2}\sum (c_1(E^{i+1}) - c_1(E^i))^2 + ...$$

In our situation $E^{r+1} = \mathbb{C}^{r+1}$ is topologically trivial, but carries a non-trivial action of $SL_{r+1}(\mathbb{C})$. Yet $c_1(E^{r+1}) = 0$, but we get an extra summand $rc_2(E^{r+1})$. This summand however is a constant $const \in H^2(BSU_{r+1})$ and will disappear from our formulas after integration over $\mathbb{C}P^1$.

Notice that $c_1(E^i) = c_1(\wedge^i E^i)$, and that the subsheaves $\wedge^i \mathcal{E}^i \subset \wedge^i \mathcal{O}^{r+1}$ are exactly those which define the map $HQ_d \to \Pi_d$. This allows us to perform our computation in the product $\mathbb{C}P^1 \times \Pi_d$ of projective spaces instead of $\mathbb{C}P^1 \times HQ_d$.

The equivariant cohomology algebra $H^*_{S^1}(\mathbb{C}P^1)$ with respect to our usual S^1-action is isomorphic to $\mathbb{Z}[\rho, \hbar]/(\rho(\rho + \hbar))$ where $\hbar = -\ln q$ is the generator of $H^*_{S^1}(pt) = H^*(BS^1)$, and $-\rho$ is the equivariant 1-st Chern class of the Hopf line bundle over $\mathbb{C}P^1$. The cohomological push-forward $pr_* : H^*_{S^1}(\mathbb{C}P^1) \to H^*_{S^1}(pt)$ is given by $pr_*(1) = 0$, $pr_*(\rho) = 1$. Computing the equivariant 1-st Chern class $c = 2\rho + \hbar$ we expand the equivariant Todd class:

$$Td(\mathbb{C}P^1) = 1 + \frac{c}{2} + \frac{c^2}{12} + ... = 1 + (\rho - \frac{\hbar}{2}) + \frac{\hbar^2}{12} + ...$$

Consider now the quot-scheme $\mathbb{P}_d := Proj(Ext^0(\mathbb{C}P^1; \mathcal{O}(-d), \mathcal{O}^N))$ corresponding to degree d maps $\mathbb{C}P^1 \to Proj(\mathbb{C}^N)$. The equivariant cohomology algebra $H^*_{S^1 \times G}(\mathbb{P}_d)$ (with respect to any linear G-action on $\mathbb{C}^N$) is generated over the coefficient ring $H^*(BS^1 \times BG)$ by the equivariant 1-st Chern class $H = c_1(P)$ of the Hopf line bundle P. The tautological composition $\mathcal{O}(-d) \otimes Hom(\mathcal{O}(-d), \mathcal{O}^N) \to \mathcal{O}^N$ defines the universal rank 1 subsheaf $\wedge \subset \mathcal{O}^N$ on $\mathbb{C}P^1 \times \mathbb{P}_d$ where therefore $\wedge = \mathcal{O}_{\mathbb{C}P^1}(-d) \otimes P$. Thus $c_1(\wedge) = H - d\rho$.

Now we apply the above construction to each factor in $\Pi_{d_1,...,d_r}$ by putting $\mathbb{C}^N = \wedge^i \mathbb{C}^{r+1}$, $d = d_i$, $i = 1, ..., r$, and conclude that the equivariant 1-st Chern classes $c_1(\wedge^i E^i)$ are represented in $\mathbb{C}P^1 \times \Pi_d$ by $H_i - d_i\rho$ where $H_i = c_1(P_i)$. We compute the degree 2 term in $Ch(V)Td(\mathbb{C}P^1)$:

$$\frac{r(r+1)}{24}\hbar^2 - (2\rho + \hbar)\sum (H_i - d_i\rho) + \frac{1}{2}\sum (H_{i+1} - H_i - (d_{i+1} - d_i)\rho)^2 + const.$$

Replacing powers of ρ in this formula by their push-forwards $pr_*(1) = 0$, $pr_*(\rho) = 1$, $pr_*(\rho^2) = pr_*(-\hbar\rho) = -\hbar$, $pr_*(const) = 0$, we find

$$c_1(\mathcal{T}_d) = -\hbar[\sum d_i + \frac{1}{2}\sum(d_{i+1} - d_i)^2] - 2\sum H_i - \sum(H_{i+1} - H_i)(d_{i+1} - d_i).$$

Since $c_1(K_d) = -c_1(\mathcal{T}_d)$, $\hbar = -c_1(q)$ and $H_i = c_1(P_i)$, we finally conclude that (at least over $\mathbb{Q}$)

$$K_d = q^{-k_d}\prod_i P_i^2(P_{i+1}P_i^{-1})^{d_{i+1}-d_i} = q^{-k_d}\prod_i P_i^{2-d_{i-1}+2d_i-d_{i+1}}.$$

3.3. Another useful property of the hyperquot schemes is that *fixed points of the action on HQ_d of the maximal torus $S^1 \times T \subset G$ are isolated.* More precisely, a fixed point of the torus $S^1 \times T$ action on HQ_d. Such a fixed point is uniquely determined by the following data:

(i) A permutation $\sigma \in S_{n+1}$ specifying a fixed point of the torus T action on the flag manifold X.

(ii) A pair Δ_+, Δ_- of lower-triangular matrices with non-negative integer entries m_{ij}, $1 \leq j \leq i \leq r$ satisfying

$$0 \leq m_{i1} \leq ... \leq m_{ii},\ i = 1, ..., r,\ \text{and}\ \sum_{i=j}^{r}(m_{ij}^+ + m_{ij}^-) = d_{r+1-j},\ j = 1, ..., r.$$

Indeed, let the flag of subsheaves $\mathcal{E}^1 \subset ... \subset \mathcal{E}^{r+1} = \mathcal{O}^{r+1}$ on $\mathbb{C}P^1$ represent a fixed point of the torus $S^1 \times T$ action on HQ_d. At generic points of $\mathbb{C}P^1$ the flag of subsheaves determines a flag of subspaces in $\mathbb{C}^{r+1}$ which may not vary along $\mathbb{C}P^1$ (S^1-invariance) and thus coincides with one of $(r+1)!$ coordinate flags in $\mathbb{C}^{r+1}$ (T-invariance). Let $(e_0, ..., e_r)$ be the standard basis in $\mathbb{C}^{r+1}$. Consider for example the T-invariant flag formed by the coordinate subspaces $\mathbb{C}^{r+1-j} := Span(e_j, ..., e_r)$ (all other T-invariant flags are obtained by permutations $(e_{\sigma(0)}, ..., e_{\sigma(r)})$ of the basis). Outside the fixed point set $(1:0), (0:1)$ of S^1-action on $\mathbb{C}P^1$ the sheaf E^{r+1-j} coincides with the sheaf of vector-functions $\mathcal{O}e_j \oplus \mathcal{O}e_{j+1} \oplus ... \oplus \mathcal{O}e_r$ (S^1-invariance). It remains to describe the $S^1 \times T$-invariant flags of subsheaves near the fixed points. It is easy to see that such a flag is equivalent to one of the following, described by the matrices Δ_+ and Δ_- near $(1:0)$ and $(0:1)$ respectively. Let ζ be the coordinate on $\mathbb{C}P^1$ near $(1:0)$, and (ζ^m) denote the ideal generated by ζ^m in the local algebra of functions on $\mathbb{C}P^1$ near at this point. In a neighborhood U of $\zeta = 0$ put

$$\mathcal{E}^{r+1-j}|_U := \oplus_{i=j}^r (\zeta^{m_{ij}})e_i\ .$$

The conditions on the matrices $\Delta_\pm$ guarantee the subsheaves form a flag and that their degrees are equal to d_{r+1-j}.

Remark. One can continue along these lines and write down explicitly Bott – Lefschetz fixed point localization formulas on HQ_d. In particular, one can easily recover this way the factorization property of the function G described by Proposition 2. On the other hand, we were unable to see how combinatorics of the localization formulas (which in principle determine the function G) implies the recursion relation (4). In the proof of Theorem 1 given in the next section we choose a different way which refers to localization formulas in the projective spaces Π_d instead. What we need to know about the hyperquot schemes is only Theorem 3 and the very fact that fixed points in HQ_d are isolated.

4. Localization and recursion

4.1. The recursion relation (4) can be restated as an identity between some elements in $K^*_{S^1\times G}(\Pi_d)$. Let

$$O_d := \mu_*(\mathcal{O}_{GX_d})$$

be the K-theoretic push-forward of the trivial line bundle over the graph space to the product Π_d of projective spaces along the map $\mu : GX_d \to \Pi_d$ described in Section 2.1. The elements O_d with different d can be compared to each other via the inclusions

$$\phi^{(i)} : \Pi_{d-\mathbf{1}_i} \subset \Pi_d$$

defined in terms of r-tuples $(f_1, ..., f_r)$ of vector-valued binary forms by the formula $f_j(x, y) \mapsto f_j(x, y)y^{\delta_{ij}}$. As it is easy to see, say, from localization to fixed points of S^1-action on Π_d, the pull-back by $\phi^{(i)}$ transforms P_j to $q^{\delta_{ij}}P_j$. Let

$$O^{(i)}_d := \phi^{(i)}_* O_{d-\mathbf{1}_i},\ i = 1, ..., r,$$

be the K-theoretic push-forward of $O_{d-\mathbf{1}_i}$ by the inclusions. One can think of O_d and $O^{(i)}_d$ as Laurent polynomials of the generators $(P_1, ..., P_r, \Lambda_0, ..., \Lambda_r, q)$ in $K^*_G(\Pi_d)$ defined modulo relations among them. Remembering the notation $Q'_i = Q_iq^{z_i}$, we get $\hat{H}_{Q',q}G(Q,Q') = \sum_d Q^d\mathcal{H}_d$ where

$$\begin{aligned}\mathcal{H}_d \ &= \ \chi_G(\Pi_d; O_d(P_1 + P_2P_1^{-1} + ... + P_r^{-1})P^z) - q^{z_1}\chi_G(\Pi_{d-\mathbf{1}_1}; O_{d-\mathbf{1}_1}P_1P^z)\\ &\quad -q^{z_2}\chi_G(\Pi_{d-\mathbf{1}_2}; O_{d-\mathbf{1}_2}P_2P_1^{-1}P^z) - ... - q^{z_r}\chi_G(\Pi_{d-\mathbf{1}_r}; O_{d-\mathbf{1}_r}P_r^{-1}P^z).\end{aligned}$$

This shows that Theorem 1 is equivalent to the following sequence of relations in $K^*_G(\Pi_d)$:

$$H_d := P_1O_d + P_2P_1^{-1}(O_d - O^{(1)}_d) + ... + P_r^{-1}(O_d - O^{(r)}_d) = (\sum \Lambda_j^{-1})O_d. \tag{6}$$

Moreover, due to the results of Section 2, the relations (6) for all $d \le d_0$ are equivalent to the recursion relations (4) for coefficients J_d of the series J for all $d \le d_0$. We are going to prove (6) by induction on $|d| := d_1 + ... + d_r$.

As we know from Section 2.3 the relation (6) is true for $|d| \le 1$.

Let us now assume that (6) is true for all $d' \ne d$ such that $d' \le d$ (componentwise). We compute the fixed point localizations of H_d at the fixed point components $\Pi^{(d^+)}_d$ of the S^1-action. More precisely, we call here the *localization* of a torus-equivariant class $A \in K^*_T(Y)$ the class a in the (localized) equivariant K-ring of the fixed point set Y^T such that $j_* a = A$ under the embedding $j : Y^T \to Y$. The localization is therefore characterized by the property $\chi_T(Y^T; aj^*B) = \chi_T(Y; AB)$ for any $B \in K_T^*(Y)$.

Let us recall from Section 2.2 that S^1-fixed points in Π_d are represented by vectors of binary forms with *monomial* components proportional to $x^{d_i^-} y^{d_i^+}$ (where $d^- = d - d^+$), that the set $\Pi_d^{(d^+)}$ formed by these fixed points is a copy of Π and that the restriction of P_i to $\Pi_d^{(d^+)}$ equals $p_i q^{d_i^+}$. Due to Proposition 1 we have, $O_d := \lambda_*(\mathcal{O}_{HQ_d}) = \mu_*(\mathcal{O}_{GX_d})$. Due to the results of Section 2 the localization of O_d to the fixed component belongs therefore to the image of $i_* : K_G^*(X) \subset K_G^*(\Pi)$ under the Plücker embedding and is described in terms of $K_G^*(X)$ by the coefficients of the series $J(Q,q) = \sum J_d(q) Q^d$ as $i_*[J_{d^+}(q) J_{d^-}(q^{-1})]$.

Similarly, localizations of $O_d^{(i)} := \phi_*^{(i)} O_{d-\mathbf{1}_i}$ coincide with $i_*[J_{d^+}(q)J_{d^-}(q^{-1})]$ since $\phi^{(i)}$ maps isomorphically the fixed point component $\Pi_{d-\mathbf{1}_i}^{(d^+ - \mathbf{1}_i)}$ onto $\Pi_d^{(d^+)}$ for all $d^+ \geq \mathbf{1}_i$. Combining, we find that localizations of H_d have the form $i_*[C_{d_+}(q) J_{d^-}(q^{-1})]$ where $C_d =$

$(p_1 q^{d_1} + \ldots + p_i p_{i-1}^{-1} q^{d_i - d_{i-1}} + \ldots + p_r^{-1} q^{-d_r}) J_d - p_2 p_1^{-1} q^{d_2 - d_1} J_{d-\mathbf{1}_1} - \ldots - p_r^{-1} q^{-d_r} J_{d-\mathbf{1}_r}$.

Comparing with the recursion relation (4) we find that the induction hypothesis implies vanishing of localizations of $H_d - (\sum \Lambda_j^{-1}) O_d$ at all fixed point component $\Pi_d^{(d^+)}$ with $d^+ \neq d$ (and the remaining vanishing condition for $d^+ = d$ coincides with (4)).

Remark. Vanishing of the localization at the last fixed set $\Pi_d^{(d)}$ will be derived with the use of the following polynomiality property. The classes H_d and O_d are defined in $K_G^*(\Pi_d)$ before of fixed point localization (while their expression in terms of $J_{d^\pm}$ makes explicit use of it) and therefore

$$\chi_G(\Pi_d; P^z[H_d - (\sum \Lambda_j^{-1}) O_d]) \ \in Repr(G)$$

is the character of a virtual representation of G and is therefore represented by a *Laurent polynomial*, that is a regular function on the maximal torus of the group $G_{\mathbb{C}} = S^1_{\mathbb{C}} \times SL_{r+1}(\mathbb{C})$. When expressed in terms of the localizations $J_{d^+}(q) J_{d^-}(q^{-1})$ (which typically have lots of other poles in q besides $q = 0, \infty$) the polynomiality property yields serious constraints on J_d. Yet it turns out (although we are not going to describe the details here) that the constraints are not powerful enough in order to uniquely determine all J_d, and we need additional geometrical information about them.

4.2. Consider further localizations J_d^σ of the coefficient $J_d(q) \in K_G^*(X)$ to the $(r+1)!$ fixed points $\sigma \in X$ of the maximal torus $T \subset SU_{r+1}(\mathbb{C})$. The restriction are rational functions of q and Λ and can be written in the form

$$J_d^\sigma = \frac{R_d^\sigma(q)}{S_d^\sigma(q)},$$

where $R_d^\sigma, S_d^\sigma \in \mathbb{Q}(\Lambda)[q]$ are polynomials in q not vanishing at $q=0$ simultaneously. We claim that in fact

$$\deg S_d^\sigma - \deg R_d^\sigma \geq k_d = d_1 + ... + d_r + \sum \frac{(d_i - d_{i-1})^2}{2} \tag{7}$$

(where we put $d_i = 0$ for $i \leq 0$ and $i > r$). This follows from general structure of Bott – Lefschetz fixed point localization formulas applied to the hyperquot schemes and from Theorem 3.

Indeed, consider the (isolated!) fixed points (σ, Δ) (see Section 3.3) in HQ_d mapped to the fixed point $\sigma \in \Pi_d^{(d)}$. As we found in Section 2.2, the coefficient J_d^σ represents the localization of $O_d = \mu_*(\mathcal{O}_{GM_d})$ at the $(r+1)!$ fixed points in $\Pi_d^{(d)}$. On the other hand we have $O_d = \lambda_*(\mathcal{O}_{QH_d})$ due to Proposition 1. Therefore the localization coefficient J_d^σ is equal to the sum

$$J_d^\sigma = \sum_\Delta \frac{1}{BL_{\sigma,\Delta}}$$

of the similar localization coefficients for $\mathcal{O}_{HQ_d}$ at the fixed points (σ, Δ) mapped to σ. The Bott – Lefschetz denominator $BL_{\sigma,\Delta}$ here is the character of the torus $S^1 \times T$-action on the exterior algebra $\wedge^* \mathcal{T}^*_{\sigma,\Delta}$ of the cotangent space to HQ_d at the fixed point. Therefore

$$BL_{\sigma,\Delta} = \Pi_\alpha (1 - q^{\nu_\alpha} \Lambda^{\chi_\alpha}).$$

where $(\nu_\alpha, \chi_\alpha)$ specify the characters of respectively S^1 and T on the 1-dimensional invariant subspaces in $\mathcal{T}^*_{\sigma,\Delta}$ indexed by α.

Each fraction $1/(1 - q^\nu \Lambda^\chi)$ with $\nu > 0$ has order 0 at $q = 0$ and order ν at $q = \infty$. Each fraction with $\nu < 0$ can be rewritten as $-q^{-\nu}\Lambda^{-\chi}/(1 - q^{-\nu}\Lambda^{-\chi})$ and has order 0 at $q = \infty$ and order $-\nu$ at $q = 0$. The product $BL^{-1}_{\sigma,\Delta}$ of such fractions is uniquely written as a rational functions in q of the form $Const\ q^M/S(q)$, where $M = -\sum \nu_\alpha$ over all negative ν_α, S is a polynomial in q, $S(0) = 1$, and the degree $\deg S = \sum_\alpha \nu_\alpha + 2M$ of the denominator exceeds the degree M of the numerator by $\sum_\alpha \nu_\alpha$ at least. Clearing denominators in the sum J_d^σ of such rational functions we represent J_d^σ as a ratio $R_d^\sigma(q)/S_d^\sigma(q)$ of two polynomials in q where

$$S_d^\sigma(0) = 1 \text{ and } \deg S_d^\sigma - \deg R_d^\sigma \geq \sum_\alpha \nu_\alpha.$$

Notice that the sum $\sum_\alpha \nu_\alpha$ coincides with the character of the S^1-action on the top exterior power of $\mathcal{T}^*_{\sigma,\Delta}$. Using Theorem 3 together with the fact that restrictions of P_i to $\Pi_d^{(d)}$ are equal to $q^{d_i} p_i$ we find

$$\sum_\alpha \nu_\alpha = -k_d + \sum d_i(2 - d_{i-1} + 2d_i - d_{i+1}) = -k_d + 2k_d = k_d.$$

regardless of the index Δ.

4.3. With the estimate (7) at hands we now complete the induction step as follows. The estimate shows that the total order of poles of J_d^σ at $q \neq 0, \infty$ is k_d at least.

On the other hand, we claim that the localizations $H_d^\sigma - (\sum \Lambda_j^{-1}) O_d^\sigma$ of $H_d - (\sum \Lambda_j^{-1}) O_d$ at the fixed points of $S^1 \times T^r$ with $d^+ = d$ and any $\sigma \in X^T$ have no poles at $q \neq 0, \infty$.

Indeed, for any $z \in \mathbb{Z}^r$ the G-equivariant holomorphic Euler characteristic of the sheaf $P^z[H_d - (\sum \Lambda_j^{-1}) O_d]$ is a Laurent polynomial of q and Λ. By the induction hypothesis localizations of this sheaf to all fixed points of S^1-action with $d^+ \neq d$ vanish, and thus the Euler characteristics is equal to

$$q^{zd} \sum_\sigma \Lambda_\sigma^z [H_d^\sigma - (\sum \Lambda_j^{-1}) O_d^\sigma],$$

The restrictions $\Lambda_\sigma^z = \Lambda_{\sigma(0)}^{-z_1} \Lambda_{\sigma(1)}^{z_1 - z_2} ... \Lambda_{\sigma(r)}^{z_r}$ of p^z to the $(r+1)!$ fixed points σ (identified in this formula with corresponding permutations) are linearly independent exponential functions of z. This shows for each σ

$$H_d^\sigma - (\sum \Lambda_j^{-1}) O_d^\sigma = \sum_{i=0}^r \Lambda_{\sigma(i)}^{-1} q^{d_{i+1} - d_i} (J_d^\sigma - J_{d-\mathbf{1}_i}^\sigma) - (\sum \Lambda_i^{-1}) J_d^\sigma, \tag{8}$$

(where we put $C_{d-\mathbf{1}_0} = 0$) must be Laurent polynomials.

Using the estimate (7) and the identities

$$k_{d-\mathbf{1}_i} - (d_{i+1} - d_i) = k_d - (d_i - d_{i-1}),$$

we see that (8) multiplied by $q^{\max_i (d_i - d_{i+1})}$ is a rational function of the form $R(q)/S(q)$ with $S(0) \neq 0$ and

$$\deg S - \deg R \geq k_d - \max(d_i - d_{i-1}) - \max(d_i - d_{i+1}).$$

This conclusion together with the property of (8) to have no finite non-zero poles leads to a contradiction unless $R = 0$ or $k_d \leq \max(d_i - d_{i-1}) + \max(d_i - d_{i+1})$. The inequality implies $|d| \leq 1$ (and turns into equality for $d = 0$ and $\mathbf{1}_i$.) Indeed, denote the maximum and the minimum by α and β. We have

$$0 \leq \alpha + \beta - k_d = \alpha + \beta - \sum \frac{(d_i - d_{i+1})^2}{2} - |d| \leq \sum \alpha - \frac{\alpha^2}{2} + \beta - \frac{\beta^2}{2} - |d| \leq 1 - |d|.$$

Thus for $d_1 + ... + d_r > 1$ we are left with the only option $R = 0$. This completes the induction step and the proof of Theorem 1.

Remarks. (1) In cohomology theory, the arguments parallel to those used in Section 4.1 (i. e. the general factorization and polynomiality properties plus explicit description of spaces of curves of degrees $|d| \leq 1$) would have been sufficient in order to determine the counterpart of the series J unambiguously and thus prove Kim's theorem [17] mentioned in the Remark (3) in the Introduction (see also Section 5.1 below). This follows from dimension counting: for $|d| > 1$ the dimension of GX_d

exceeds the degree of the cohomology class analogous to $H_d - (\sum \Lambda_j^{-1}) O_d$ by a margin large enough in order to provide the necessary number of additional linear dependencies among fixed point localizations.

(2) In K-theory, the dimensional argument is not available. Let us look however at the integral formula (5) (yet conjecturally) representing $\mathcal{G}$ for $r = 2$. For d_1, d_2 large enough and $z_1, z_2 > 0$ small enough the integrand has no poles with $P_1, P_2 = 0$ or ∞, and thus the integral is equal to 0. If known *a priori*, this property would provide enough linear dependencies between localizations of H_d in order to complete our proof.

(3) The same property of the countur integral (see Section 2.4) representing the Bott – Lefschetz formula for complex projective spaces follows *a priori* from the Kodaira – Nakano vanishing theorem. It would be interesting to figure out what kind of vanishing theorems would guarantee this property in the case of the graph spaces GX_d.

(4) In the above proof we exploited another property which is manifest in (5): the difference $d_1^2/2 + (d_1 - d_2)^2/2 + d_2^2/2 + d_1 + d_2$ between the degrees of the numerator and the denominator in the integrand considered as a function of q^{-1}. The margin is explained by Theorem 3, and the whole argument resembles the famous proof [1] of Atiyah – Hirzebruch rigidity theorem of arithmetical genus refined by the estimate of the equivariant canonical class.

5. Generalization to flag manifolds G/B

An arbitrary complex semi-simple Lie group G will replace here the group $SL_{r+1}(\mathbb{C})$ of the previous sections.

5.1. In order to generalize Theorems 1 and 2 to flag manifolds G/B of semi-simple complex Lie algebras $\mathfrak{g}$ it is useful to recall corresponding results of quantum cohomology theory due mainly to B. Kim [17]. According to the Borel – Weil construction, fundamental representations $V_1, ..., V_r$ of the Lie algebra $\mathfrak{g}$ of rank r can be realized in the spaces of holomorphic sections of suitable line bundles over G/B with the 1-st Chern classes $h_1, ..., h_r$. The sections define the *Plücker embedding*

$$X := G/B \to \Pi := \Pi_{i=1}^r Proj(V_i^*),$$

which allows one to generalize the construction of the maps μ graph spaces from GX_d to the products Π_d of projective spaces. The cohomological counterpart of the series (1) is

$$\mathcal{G}^H = \sum_{d=(d_1,...,d_r)} Q_1^{d_1} ... Q_r^{d_r} \int_{GX_d} e^{H_1 z_1 + ... + H_r z_r},$$

where H_i are $S^1 \times G$-equivariant 1-st Chern classes of the hyperplane line bundles on the product Π_d of projective spaces pulled back to the graph space GX_d by the map μ. The power (Q, z)-series $\mathcal{G}_H$ with coefficients in $H^*(BS^1 \times BG, \mathbb{Q})$

can be factored as $(I_H(Qe^{\hbar z}, -\hbar^{-1}), I_H(Q,\hbar^{-1}))$ where $(.\,,.)$ is the G-equivariant Poincaré pairing on $H^*_G(X)$, $I_H(Q,\hbar^{-1}) = e^{(h \ln Q)/\hbar}[\sum_d c_d Q^d]$ is a suitable series with coefficients $c_d \in H^*_G(X, \mathbb{Q}(\hbar))$, and $\hbar$ is the generator of $H^*(BS^1)$.

The main theorem in [17] says that I_H *is a common eigen-function of* r *commuting differential operators which form a complete set of quantum conservation laws of the quantum Toda lattice corresponding to the Langlands-dual Lie algebra* $\mathfrak{g}'$.

For example, the Hamiltonian operator of the Toda lattice in the (self-dual) sl_{r+1}-case has the form

$$H = \frac{\hbar^2}{2}\sum_{i=0}^{r}\frac{\partial^2}{\partial t_i^2} - \sum_{i=1}^{r} e^{t_{i-1}-t_i},$$

and $H\ I = \frac{1}{2}(\lambda_0^2 + ... + \lambda_r^2)\ I$ (we identify here $H^*(BG)$ with the algebra of symmetric functions in $\lambda_0, ..., \lambda_r$, $\sum \lambda_i = 0$). As we mentioned in the Remark (2) in the Introduction, the equation for I_H can be extracted from the finite-difference equation $\hat{H} I = (\sum \Lambda_i^{-1}) I$ as the degree 2 part in the following approximation: put $q = e^{-\hbar} = 1 - \hbar + ...$, $\Lambda_i = e^{\lambda_i} = 1 + \lambda_i + ...$ and use the grading $\deg \hbar = 1, \deg \lambda_i = 1, \deg Q_i = 2$.

In K-theory, we introduce

$$\mathcal{G}_K := \sum_{d=(d_1,\ldots,d_r)} Q_1^{d_1}...Q_r^{d_r}\chi_{S^1\times G}(GX_d; P_1^{z_1}...P_r^{z_r}),$$

where P_i denote the Hopf line bundles on Π_d pulled back to GX_d by μ. Due to the factorization property of Section 2 we have (for $Q' = Qq^z$):

$$\mathcal{G}_K = \langle I_K(Q', q), I_K(Q, q^{-1})\rangle$$

where the series $I_K(Q,q) = p^{\ln Q/\ln q}\sum J_d Q^d$ has coefficients $J_d \in K^*_{S^1\times G}(X)$. The conjecture we are about to describe says that I_K *is a common eigen-function of the conservation laws of the finite-difference Toda lattice corresponding to* $\mathfrak{g}'$.

5.2. The commuting differential operators of quantum Toda lattices originate (see [19, 24]) from the center $Z_{U\mathfrak{g}'}$ of the universal enveloping algebra $U\mathfrak{g}'$. Similarly, commuting finite-difference operators of the Toda lattice originate from the center of the corresponding quantum group $U_q\mathfrak{g}'$.

Let $\mathfrak{g}'$ be the simple complex Lie algebra Langlands-dual to $\mathfrak{g}$ (i. e. $\mathfrak{g}' = \mathfrak{g}$ unless they have the types B_r and C_r which are dual to one another). The quantum group $U_q\mathfrak{g}'$ is defined as a (Hopf) algebra in terms of generators $K_i^{\pm 1}, X_i^{\pm}, i = 1, ..., r$ and relations (see for instance [23]):

$$[K_i, K_j] = 0,\ K_i X_j^{\pm} = q_i^{\pm a_{ij}} X_j^{\pm} K_i,\ [X_i^+, X_j^-] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i^2} q_i^{-k(1-a_{ij}-k)}(X_i^{\pm})^k X_j^{\pm}(X_i^{\pm})^{1-a_{ij}-k}=0 \quad \text{for } i\neq j,$$

where $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$ is the Cartan matrix of $\mathfrak{g}'$, $\alpha_1,...,\alpha_r$ — simple roots of $\mathfrak{g}'$, $(.,.)$ — a W-invariant inner product, $q_i=q^{(\alpha_i,\alpha_i)/2}$, and $\binom{m}{k}_q$ is the q-binomial coefficient

$$\frac{(1-q)(1-q^2)...(1-q^m)}{(1-q)...(1-q^k)\ (1-q)...(1-q^{m-k})}.$$

The Cartan subalgebra $U_q\mathfrak{h}'=\mathbb{Q}[K^{\pm 1}]$ is the group algebra of the root lattice for $\mathfrak{g}'$. In the Toda theory it is useful to extend the root lattice (and the quantum group) to the *co*-weight lattice of $\mathfrak{g}'$. We introduce the commuting generators $P_1,...,P_r$ corresponding to fundamental weights of $\mathfrak{g}$ (i.e. to fundamental co-weights of $\mathfrak{g}'$) so that $K_i=P_1^{(\alpha_i,\alpha_1)}...P_r^{(\alpha_i,\alpha_r)}$ and choose new generators $Q_i^{\pm}=X_i^{\pm}P_1^{\pm m_{i1}}...P_r^{\pm m_{ir}}$ instead of $X_i^{\pm}$. Here (m_{ij}) is any matrix with the property

$$m_{ij}=m_{ji} \text{ unless } (\alpha_i,\alpha_j)<0 \text{ in which case } m_{ji}-m_{ij}=\pm(\alpha,\alpha)/2\ ,$$

where α is a longest of the two roots. (For instance, one can orient the edges of the Dynkin diagram and put $m_{ji}=0$ unless there is an edge from i to j in which case put $m_{ji}=(\alpha,\alpha)/2$.) The relations between the new generators read:

$$[P_i,P_j]=0,\ P_iQ_j^{\pm}=q^{\pm\delta_{ij}}Q_j^{\pm}P_i,\ Q_i^+Q_j^- - q^{m_{ji}-m_{ij}}Q_j^-Q_i^+=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},$$

and for $i\neq j$, $x:=q_i^{a_{ij}\pm 2(m_{ji}-m_{ij})/(\alpha_i,\alpha_i)}$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i^2}(q_i^2)^{k(k-1)/2}\ x^k\ (Q_i^{\pm})^kQ_j^{\pm}(Q_i^{\pm})^{1-a_{ij}-k}=0\ . \tag{9}$$

Notice that the Serre relations (9) allow 1-dimensional representations $\chi_-:U_q\mathfrak{n}'_-\to\mathbb{C}$ of the subalgebra generated by $(Q_1^-,...,Q_r^-)$ due to the following q-binomial identity:

$$(1-x)(1-qx)...(1-q^{m-1}x)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}_q q^{k(k-1)/2}x^k\ ,$$

and that the Borel subalgebra $U_q\mathfrak{b}'_+$ generated by $P_i^{\pm 1}$ and Q_j^+ has a representation π_+ onto the algebra of finite-difference operators $Q_i^+\mapsto Q_i\times,\ P_i\mapsto q^{Q_i\partial/\partial Q_i},\ i=1,...,r$.

In order to construct commuting finite-difference operators from the center $Z_{U_q\mathfrak{g}'}$ of the quantum group we, following B. Kostant, represent the quantum group as the tensor product $U_q\mathfrak{b}'_+\cdot U_q\mathfrak{n}'_-$ of the subspaces, then notice that the linear map $U_q\mathfrak{g}'\to U_q\mathfrak{b}'_+\otimes U_q\mathfrak{n}'_-$ restricted to the center is a homomorphism of algebras $Z_{U_q\mathfrak{g}'}\to U_q\mathfrak{b}'_+\otimes U_q^{\circ}\mathfrak{n}'_-$ (here $^{\circ}$ means the anti-isomorphic algebra) and compose the homomorphism with the representation $\pi_+\otimes\chi_-$.

5.3. The center $Z_{U_q\mathfrak{g}'}$ for generic q is known to be isomorphic to the polynomial algebra in r generators. The corresponding finite-difference operators one obtains by choosing $\chi_- = 0$ are constant coefficient linear combinations of translation operators. Considered as Laurent polynomials of the elementary translations $\hat{P}_i = q^{Q_i\partial/\partial Q_i}$, they become W-invariant after the ρ-shift

$$\hat{P}_i \mapsto \hat{P}_i q^{-\rho_i} = Q^\rho \hat{P}_i Q^{-\rho},$$

where $\sum \rho_i\alpha_i = \rho$ is the semi-sum of positive roots of $\mathfrak{g}'$. The W-invariance follows from the theory of Verma modules for the quantum group $U_q\mathfrak{g}'$.

One defines commuting operators $\hat{D}_i,\ i = 1, ..., r$, of the finite-difference Toda lattice by choosing a generic character on the role of χ_- (i. e. $\chi_-(Q_j^-) \neq 0$ for all j) and applying the above construction followed by the ρ-shift to generators $D_1, ..., D_r$ of the center. Then the constant coefficients symbols $Smb_D := \hat{D}_i$ mod (Q) are W-invariant Laurent polynomials in $(\hat{P}_1, ..., \hat{P}_r)$. Recall now that the generators P_i correspond to fundamental weights of $\mathfrak{g}$. This allows us to consider the symbols as W-invariant functions on the maximal torus of G. Finally, the form of the finite-difference Toda system we need reads:

$$\hat{D}_i\ I = Smb_{D_i}(\Lambda)\ I, \quad i = 1, ..., r, \tag{10}$$

where $\Lambda = (\Lambda_1, ..., \Lambda_r)$ are Laurent coordinates on the maximal torus of G equal to highest weights of the fundamental representations.

Conjecture. *The $K^*_{S^1\times G}(X)$-valued formal function $I_K(Q, q)$ (and respectively the generating function $\mathcal{G}_K = \langle I_K(Qq^z, q), I_K(Q, q^{-1})\rangle$ for the flag manifold $X = G/B$ satisfies the system (10) of finite-difference Toda equations corresponding to the Langlands-dual Lie algebra $\mathfrak{g}'$.*

5.4. Remarks. (1) Operators of the Toda system (10) depend on the choice of the matrix $(m_{ji} - m_{ij})$ and of the generic character χ_-. The latter ambiguity is compensated by rescalings of $Q_1, ..., Q_r$. A canonical choice of the matrix for the classical series A_r, B_r, C_r, D_r is described in [23] (Theorem 12). At the moment we are not ready to specify the choice to be made in the above Conjecture.

(2) The construction of commuting Toda operators is an algebraic version of the following geometrical description for differential Toda systems: interpret the center $Z_{U\mathfrak{g}'}$ of the universal enveloping algebra as the algebra of bi-invariant differential operators on the group G' and make them act on the functions on the maximal torus T' by extending such functions to the dense subset $N'_+T'N'_- \subset G'$ equivariantly with respect to given generic characters $\chi_\pm : N'_\pm \to \mathbb{C}^\times$ (i. e. restrict the operators to the sheaf of functions f on G' satisfying $f(n_+gn_-^{-1}) = \chi_+(n_+)f(g)\chi_-(n_-^{-1})$). The Hamiltonian differential operator H of the quantum Toda system is obtained by this construction (followed by the ρ-shift) from the bi-invariant Laplacian on G'.

(3) By the Harish-Chandra isomorphism, central elements in $U\mathfrak{g}'$ correspond to ad-invariant elements in $S(\mathfrak{g}')$ (or to W-invariant polynomials on the Cartan subalgebra of $\mathfrak{g}$). Similarly, central elements in $U_q\mathfrak{g}'$ correspond to W-invariant functions on the maximal torus of G. This explains why the operator (2) does not look like a finite-difference version of the 2-nd order differential operator H: the operator $\hat{H}$ corresponds to the trace of unimodular matrices, while H corresponds to the trace of square for traceless matrices. For classical series there are relatively simple algebraic formulas for generators of $Z_{U_q\mathfrak{g}'}$ quantizing traces of powers of matrices in the vector representation (see [23], Theorem 14). It is not hard to point out explicitly the finite-difference Toda operator corresponding in the above Conjecture to the trace in the *co*-vector representation of a classical group.

5.5. Whittaker functions. In harmonic analysis, one constructs eigen-functions of the center $Z_{\mathfrak{g}'}$ as matrix elements of irreducible representations of G'. This construction in the case of the Toda system (10) includes the following ingredients. Let $|v\rangle \in V$ be a *Whittaker vector* in an irreducible representation of $U_q\mathfrak{g}'$, i. e. a common eigen-vector of the elements Q_i^-, $i = 1, ...r$: $Q_i^-|v\rangle = \chi_-(Q_i^-)|v\rangle$. Let $\langle a| \in V^*$ be a *Whittaker covector*: $\langle a|Q_i^+|x\rangle = \chi_+(Q_i^+)\langle a|x\rangle$ for any $|x\rangle \in V$, $i = 1, ..., r$. Let $Smb_D(\Lambda q^\rho)$ be the eigenvalue of the central element D in the representation V. Then the matrix element

$$\langle a|P_1^{\ln Q_1/\ln q} ... P_r^{\ln Q_r/\ln q}|v\rangle,$$

up to the ρ-shift, is an eigenfunction of the finite-difference operator $\hat{D}$ with the eigenvalue Smb_D.

In the case of differential Toda lattices the construction of eigen-functions as matrix elements (which requires existence of suitable Whittaker vectors, covectors and integrability of the representation of $\mathfrak{g}'$ to the maximal torus $T' \subset G'$ at least) can be realized (see [19, 24]) in suitable analytic versions of the principal series representations on the role of V. According to [7, 25] (see the Remark (5) in the Introduction) the construction can be carried over to the case of quantum groups. We arrive therefore at the following representation-theoretic interpretation of the Conjecture

$$\begin{array}{c}\text{generating functions for}\\ \text{representations of } S^1\times G\\ \text{in cohomology of line bundles on}\\ \text{spaces of rational curves in } G/B\end{array} = \begin{array}{c}\text{matrix elements in}\\ \infty\text{-dimensional representations}\\ \text{of the Langlands-dual}\\ \text{quantum group } U_q\mathfrak{g}'\end{array}.$$

We expect that the LHS of this equality has a natural interpretation in terms of representation theory for the Kac-Moody loop group $\hat{LG}$ at the critical level (rather than in terms of generating functions for representations of $S^1 \times G$). The reason is that the graph spaces GX_d (or the quasimap spaces $\mu(GX_d) \subset \Pi_d$) can be considered as degree d approximations to the loop space LX with the circle action induced by the rotation of loops. Moreover, according to [14] these spaces provide adequate geometrical background for semi-infinite representation theory of loop groups. However we do not have at the moment any direct evidence in favor of this expectation.

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