Math 53. Multivariable Calculus. Midterm II. 11.06.15
Solutions

1. Find a value of k for which there exists a function $f(x, y)$ such that $f_x = kx + 6y$, $f_y = kx - 6y$, and find such a function.

From Clairaut's theorem, $k = f_{yx} = f_{xy} = 6$, and so $f = 3x^2 + 6xy - 3y^2$ will do.

2. For each critical points of the function $x^2y - y + y^3/3$, determine whether it is a local maximum, a local minimum, or a saddle.

From $f_x = 2xy = 0$ and $f_y = x^2 + y^2 - 1 = 0$ we find four critical points $(x, y) = (\pm 1, 0)$ and $(0, \pm 1)$. The first two are saddles, since $D := f_{xx}f_{yy} - f_{xy}^2 = (2y)(2y) - (2x)^2 = 4(y^2 - x^2)$ at these points is negative. At $(0, 1)$ and $(0, -1)$ the function has a local minimum and maximum respectively, since $D > 0$ at both, while $f_{xx} > 0$ at the former and $f_{xx} < 0$ at the latter.

3. Find possible values at the point $(x, y) = (1, 2)$ of a differentiable function $z(x, y)$ implicitly defined by the equation $x^2+2xy+3yz+2z^2 = 5$, and compute the gradient vector of each branch of this functions at this point.

At $(x, y) = (1, 2)$ we find $1^2 + 2^2 + 6z + 2z^2 = 5$, i.e. $3z + z^2 = 0$, and hence possible values of z are 0 and -3. From

$$(2x + 2y)dx + (2x + 3z)dy + (3y + 4z)dz = 0$$

we find $z_x = -(2x+2y)/(3y+4z)$ and $z_y = -(2x+3z)/(3y+4z)$. For $(x, y, z) = (1, 2, 0)$, and $(x, y, z) = (1, 2, -3)$ this yields $\nabla z(x, y) = -\mathbf{i} - \mathbf{j}/3$, and $\nabla z(x, y) = \mathbf{i} - 7\mathbf{j}/6$ respectively.

4. Find the maximum and minimum values of the function $f(x, y) = x$ in the region $2x^2 + 6xy + 9y^2 \leq 9$.

Since x has no critical points, the extremal values in the region bounded by the ellipse $2x^2+6xy+9y^2 = 9$ are achieved at the boundary. By the Lagrange multiplier method, we find the constrained extrema from

$$1 = \lambda(4x + 6y),\ 0 = \lambda(6x + 18y),\ 2x^2 + 6xy + 9y^2 = 9.$$

From the 1st equation, $\lambda \neq 0$, from the 2nd, $y = -x/3$, and from the 3rd we have $2x^2 - 6x^2/3 + 9x^2/9 = 9$, i.e. $x^2 = 9$. Thus, $x = 3$ and $x = -3$ are the maximum and minimum values.

5. Verify that for arbitrary twice differentiable functions f and g in one variable each, the function $u(x,t) = f(x-ct) + g(x+ct)$ satisfies the so called *wave equation* $u_{tt} = c^2 u_{xx}$ (where c is a given constant).

We have: $u_x = f'(x-ct) + g'(x+ct)$,
and hence $u_{xx} = f''(x-ct) + g''(x+ct)$,
while $u_t = f'(x-ct)(-c) + g'(x+ct)c$,
and so $u_{tt} = f''(x-ct)(-c)^2 + g''(x+ct)c^2 = c^2 u_{xx}$

6. Give an example of a two-times differentiable function $f(x,y)$ which has a critical point at the origin $(x,y) = (0,0)$, and satisfies $f_{xx}(0,0) = 4$, $f_{xy}(0,0) = 6$, $f_{yy}(0,0) = 9$, but does not have a local minimum at the origin.

By Taylor's formula, near the critical point $(0,0)$ we have

$$f(x,y) = f(0,0) + \frac{1}{2}\left(4x^2 + 6xy + 9y^2\right) + o(x^2+y^2).$$

Since $D = 4 \cdot 9 - 6^2 = 0$, the quadratic part is the square of a linear function (in fact it is $(2x+3y)^2/2$). Thus it vanishes on a line passing through the origin (in fact on the line $2x + 3y = 0$), and adding any higher order terms, negative everywhere on this line but the origin, will do. For instance, taking $o(x^2+y^2) = -(x^2+y^2)^2$ or $-(x^4+y^4)$ will suffice.