Practice problems from past exams

1. Change the order of integration in the triple integral:

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\ dz\ dy\ dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\ dy\ dx\ dz\ .$$

2. Compute the volume of the unit ball $x^2+y^2+z^2+t^2 \le 1$ in the 4-dimensional space.

3. Does there exist a function f whose partial derivatives are:

(a) $f_x = y\sin xy, \quad f_y = x\sin xy$?

(b) $f_x = x\sin xy, \quad f_y = y\sin xy$?

4. The graph of a function $z = f(x,y)$ is the surface of revolution obtained by rotating the parabola $z = x^2/2$, lying in the plane $y = 0$, about the z-axis. Compute the directional derivative of this function at the point $(x,y) = (10,25)$ in the direction of the vector $(0,-1)$.

5. Express the derivative $\partial^2 f/\partial u\partial v$ of the composite function obtained from a twice differentiable function $f(x,y)$ by the substitution $x = u+v$, $y = uv$.

6. Give an example of a function f with $f_{xy} \ne f_{yx}$.

7. Find all differentiable functions $f(x,y)$ such that $f_x = y\exp xy$.

8. Compute the gradient of the function $1/|\vec{r}|$.

9. Find all values and partial derivatives at the point $(a,b) = (5,4)$ of the function $x(a,b)$ defined implicitly by the equation $x^4 - ax^2 + b = 0$.

10. Find and classify all critical points of the function $f = xy - x^3/3 - y^3/3$.

11. Find the global maximum and minimum values of the function $\sqrt{x^2+y^2}$ in the region

$$-1 \le x \le 3, \qquad -2 \le y \le 4.$$

12. Find all critical points and critical values of the function $x+y+z$ under the constraint $2xy^2z^3 = 1$.

The angular momentum vector $\vec{M}$ of a moving particle with respect to a center is defined as $\vec{M} = m(\dot{\vec{r}} \times \vec{r})$, where $\vec{r}$ is the radius-vector of the particle with respect to the center, $\dot{\vec{r}}$ is the velocity vector, and m is the mass of the particle.

13. Derive the angular momentum conservation law: If a particle moves according to Newton's 2nd law "$F = ma$" in any force field directed to a center, then the angular momentum vector with respect to this center does not change.

14. Astronomers observe a strange planet, Swingus, moving around the Sun $(0,0,0)$ according to the law: $x = a\cos t,\ y = b\sin t,\ z = 0$. Describe the orbit of Swingus, and find out if the motion obeys the angular momentum conservation law.