Practice problems from past exams

1. A scalar-valued function f on the plane is called *differentiable at a point* $\vec{r}$ if there exists a linear function ϕ such that:

$$\lim_{\vec{u}\to\vec{0}} \frac{f(\vec{r}+\vec{u})-f(\vec{r})-\phi(\vec{u})}{|\vec{u}|}=0.$$

Which of the following statements are true?

The function on the plane defined as the distance to the origin is differentiable at the origin.

Every function all of whose directional derivatives at $\vec{r}$ exist is differentiable at $\vec{r}$.

Every function whose both partial derivatives at $\vec{r}$ exist is continuous at $\vec{r}$.

The function $f(x,y)$ equal to 0 at $(x,y)=(0,0)$ and to $xy(x+y)/\sqrt{x^2+y^2}$ at $(x,y)\neq(0,0)$ is differentiable at $(x,y)=(0,0)$.

The function $f(x,y)$ equal to 0 at $(x,y)=(0,0)$ and to $xy/(x^2+y^2)$ at $(x,y)\neq(0,0)$ is continuous at $(x,y)=(0,0)$.

2. Does there exist a function f whose partial derivatives are:

(a) $f_x = y\sin xy,\quad f_y = x\sin xy$?

(b) $f_x = x\sin xy,\quad f_y = y\sin xy$?

3. The graph of a function $z=f(x,y)$ is the surface of revolution obtained by rotating the parabola $z=x^2/2$, lying in the plane $y=0$, about the z-axis. Compute the directional derivative of this function at the point $(x,y)=(10,25)$ in the direction of the vector $(0,-1)$.

4. Express the derivative $\partial^2 f/\partial u\partial v$ of the composite function obtained from a twice differentiable function $f(x,y)$ by the substitution $x=u+v$, $y=uv$.

5. Give an example of a function f with $f_{xy}\neq f_{yx}$.

6. Find all differentiable functions $f(x,y)$ such that $f_x = y\exp xy$.

7. Compute the gradient of the function $1/|\vec{r}|$.

8. Find all values and partial derivatives at the point $(a,b)=(5,4)$ of the function $x(a,b)$ defined implicitly by the equation $x^4-ax^2+b=0$.

9. Which of the conclusions:

(a) f has a local minimum;
(b) f has neither a local minimum nor a local maximum;
(c) f does not have a local minimum but is not guaranteed to have a local maximum;
(d) f does not have a local maximum but is not guaranteed to have a local minimum

can be made about the behavior of a function $f(x,y)$ near the origin:

A. $f=9x^2-6xy+y^2+o(x^2+y^2)$ B. $f=9x^2-5xy+y^2+o(x^2+y^2)$ C. $f=9x^2+5xy-y^2+o(x^2+y^2)$ D. $f=9x^2+6xy+y^2+o(x^2+y^2)$ E. $f=8x^2+6xy+y^2+o(x^2+y^2)$.

10. What should be the ratio between the radius r and the height h of a cylinder of a given volume so that its total surface area is minimal?

11. Find and classify all critical points of the function $f = xy - x^3/3 - y^3/3$.

12. Find the global maximum and minimum values of the function $\sqrt{x^2+y^2}$ in the region

$$-1 \le x \le 3, \qquad -2 \le y \le 4.$$

13. Find all critical points and critical values of the function $x+y+z$ under the constraint $2xy^2z^3 = 1$.

14. Which of the following statements about the function $f = 3x^2 - 2y^2 + z^2$ in the region $x^2+y^2+z^2 \le 1$ are true?

- f does not have critical points in the interior of the region.
- The maximum value of f is achieved on the boundary of the region.
- The minimum value of f is achieved on the boundary of the region.
- Critical points of f *restricted* to the boundary of the region are those points of the boundary where the gradient vector of f is proportional to the radius-vector $\vec{r}$.
- Ctitical values of f restricted to the boundary coincide with critical values of

$$F(x, y, z, \lambda) = 3x^2 - 2x + z^2 + \lambda(x^2 + y^2 + z^2) - \lambda.$$

- The number of critical points of f restricted to the boundary is < 6.
- The number of different critical values of f restricted to the boundary is > 3.