Mock Putnam 2018 (UC Berkeley)

1. Find the distance between the graphs of the functions

$$y = e^{2018x} \text{ and } y = \frac{1}{2018}\ln x.$$

2. Let p be a prime number. Prove that the numerator m of the fraction

$$\frac{m}{n} := 1 + \frac{1}{2} + \cdots + \frac{1}{p-1}$$

is divisible by p.

3. On each side of n cards, one of the numbers $1, 2, \ldots, n$ is written in such a way that overall each of the numbers occurs exactly twice. Prove that the cards can be laid out on the table in such a way that all numbers from 1 through n will be on the top sides.

4. A code lock is organized as a 4×4 switch board. Changing a switch from "up" to "down" position or vice versa automatically reverses positions of all the other six switches in the same row and column as the first one. In the unlocked position, all switches are up. Find an algorithm of unlocking the code lock starting from any given initial configuration of the switches.

5. In the linear space $C[0,1]$ of all real-valued continuous functions on $[0,1]$, the unit cube K is defined as the set of those functions whose maximal absolute value equals 1:

$$K = \{f : [0,1] \to \mathbb{R} \mid \max_{x\in[0,1]} |f(x)| = 1\}.$$

Show that $C[0,1]$ has a subspace, whose intersection with K is (a) a circle; (b) an icosahedron; (c) a sphere.

6. Show that for any positive integer n there exists a 2018×2018-matrix A such that

$$A^n = \begin{bmatrix} 1 & 2 & 3 & \ldots & 2017 & 2018 \\ 0 & 1 & 2 & 3 & \ldots & 2017 \\ 0 & 0 & 1 & 2 & \ldots & 2016 \\ & \ldots & & & \ldots & \\ 0 & 0 & \ldots & & 1 & 2 \\ 0 & 0 & 0 & \ldots & & 1 \end{bmatrix}.$$