Mock Putnam Competition – October 4, 2013

1. Find the greatest integer, not divisible by 10, which is a complete square and such that after crossing out its two rightmost digits, the remaining number is also a complete square.

2. How many real roots does the following polynomial have:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} \quad ?$$

3. Prove that for every positive integer n, equations $x^2 + y^2 = n$ and $x^2 + y^2 = 2n$ have the same number of integer solutions (x, y).

4. Let $a_1, a_2, \ldots, a_n, \ldots$ be a bounded sequence of positive integers. It is known that

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = 1.$$

Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 1.$$

5. Prove that system of equations

$$\begin{aligned} p_1 z_1 + p_2 z_2 + \cdots + p_n z_n &= 0 \\ p_1 z_1^2 + p_2 z_2^2 + \cdots + p_n z_n^2 &= 0 \\ &\cdots \\ p_1 z_1^n + p_2 z_2^n + \cdots + p_n z_n^n &= 0 \end{aligned}$$

where $p_1, ..., p_n$ are given positive real numbers, has a unique complex solution $z_1 = z_2 = \cdots = z_n = 0$.

6. Let $B = C^{-1}AC$, where A and C are $n \times n$-matrices with integer entries, $\det A = 1$, and $\det C \neq 0$. Prove that there exists a positive integer m such that all entries of B^m are integer.