

Mock Putnam 2016 (UC Berkeley)

**1.** Three pedestrians are travelling with constant speeds along three straight roads. Show that if at the starting moment they were not on the same line, then there are at most two moments of time when they can turn out on the same line.

**2.** Let  $P(x)$  be a polynomial with non-negative integer coefficients, and  $p_n$  denote the sum of the digits of  $P(n)$ . Show that there exists an integer which occurs in the sequence  $p_1, p_2, p_3, \dots$  infinitely many times.

**3.** Given a sequence  $\{a_n\}$  of real numbers such that

$$\lim_{n \rightarrow \infty} \left( a_{n+1} - \frac{a_n}{2} \right) = 0.$$

Prove that  $\lim_{n \rightarrow \infty} a_n = 0$ .

**4.** What is the smallest perimeter that a strictly convex 32-gon with integer vertices can have?

**5.** Let  $A$  be an invertible  $n \times n$ -matrix such that in each row of  $A$  one entry is equal to  $\pm 1$  while all others are 0. Prove that there exists a positive integer  $k$  such that  $A^k = A^T$  (“A-transposed”).

**6.** A sequence of positive integers is defined by  $a_1 = 1$ , and  $a_{n+1} = a_n^3 + 1$  for each  $n \geq 1$ . Find the greatest common divisor of  $a_{100}$  and  $a_{2016}$ .