Answers to HW9

3.3.10. Let z be a regular value of $g \circ f$. Then each $y \in g^{-1}(z)$ must be regular values of f, and for each $x \in f^{-1}(y)$, we have $\operatorname{sign} d_x(g \circ f) = (\operatorname{sign} d_x f)(\operatorname{sign} d_{f(x)} g)$. Thus

$$\deg(g \circ f) = \sum_{y \in g^{-1}(z)} \operatorname{sign} d_y g \left(\sum_{x \in f^{-1}(y)} \operatorname{sign} d_x f \right) = (\deg g)(\deg f),$$

since for each y the interior sum equals $\deg f$.

2 (= 3.3.20). It was mostly done in the previous hw in terms of Lefschetz' fixed point theory. Likewise, in terms of Hopf-Poincaré's theory, the Euler characteristic of a manifold X can be defined as the self-intersection index $I(Z, Z)$ of the zero section Z in TX. The tangent bundle space TX is orientable even when X is not, in which case the intersection index can be redefined as $(-1)^{\dim X} I(Z \times Z, \Delta_{TX})$, where Δ_{TX} is the diagonal in $TX \times TX$. The point is the latter intersections occur in a vicinity of the diagonal $\Delta_Z \subset Z \times Z$, which (the vicinity) is orientable, being diffeomorphic to TZ again.

3. A vector field v on $\mathbb{R}P^n$ with non-degenerate zeroes lifts to the double cover S^n as vector field $\tilde{v}$ with non-degenerate zeroes, two such zeroes per every zero of v, both contributing the same sign into the Hopf-Poincaré index sum. Thus, $\chi(\mathbb{R}P^n) = \chi(S^n)/2 = 1$ for even n and 0 for odd.

3.4.10. (a) For $f(z) = z + z^m$, on the circle $z = \epsilon e^{it}$, $(f(z) - z)/|f(z) - z| = e^{imt}$, which has degree m as a map from S^1 to S^1.

(b) The peturbed map $f_c(z) = c + z + z^m$ has m fixed points $z = (-c)^{1/m}$, non-degenerate since at these points $f_c' = 1 + mz^{m-1} \neq 1$, all approaching $z = 0$ as $c \to 0$, and each contributing $+1 = \operatorname{sign} |f_c'((-c)^{1/m})|^2$ into the Lefschetz number.

(c) At $z = \epsilon e^{it}$, we have $(z + \bar{z}^m - z)/|z + \bar{z}^m - z| = e^{-imt}$ which has degree $-m$ as a map $S^1 \to S^1$.

3.5.18. The hint works indeed: The degree of the map $v/|v| : \partial W \to S^{k-1}$ equals the sum of the degrees of such maps defined on the boundary spheres of small balls around isolated zeroes of v inside W, because the map is well-defined on the entire complement to these balls in W.