Answers to HW8

3.2.13. Yes, as the hint suggests, by the Jordan-Brouwer separation theorem, each connected component of the hypersurface is two-sides, i.e. its normal 1-dimensional bundle is orientable, and since the ambient Euclidean space is also orientable, the tangent bundle to the hypersurface is orientable as well.

3.2.23. As some of you noticed, the claim is false, unless X is connected. If it is, then X has only two orientations, and the reversal of one on X results in no change of the product orientation on $X \times X$ (in the end, because $-$ times $-$ is $+$).

3.3.2. (a) On the standard $S^k \subset \mathbb{R}^{k+1}$, the antipodal map maps e_0 to $-e_0$, which can be rotated back into e_0 inside the plane $Span(e_0, e_1)$ around the subspace $Span(e_2, \ldots, e_k)$ (whose points remain fixed in the process the rotations). The basis $e_1, \ldots e_k$ in tangent space $T_{e_0}S^k$ is mapped to $-e_1, \ldots, -e_k$ under the antipodal map, and then to $e_1, -e_2, \ldots, -e_k$ under the 180-degree rotation. Thus, the degree of the antipodal map equals $(-1)^{k-1}$. [Another argument: the map $x \mapsto -x$ changes/preserves the orientation of $\mathbb{R}^{k+1}$ according to the sign $(-1)^{k+1}$, but preserves the exterior normal direction to the sphere.]

(b) Since the degree of the identity map is 1, when $k+1$ is odd, it is not homotopic to the antipodal map (whose degree in this case is -1). When $k+1$ is even, $z \mapsto e^{\pi i t} z$ provides the homotopy on $S^k = \{z \in \mathbf{C}^{(k+1)/2} \mid |z| = 1\}$.

(c) According to some old exercise, when a sphere has a non-vanishing vector field, the antipodal map is homotopic to the identity. (To remind: one can move each point with speed 1 along the great circle in the direction of the vector during time π.) By (b), a vector field cannot exist on even-dimensional spheres, and on odd-dimensional ones, the velocity vector field of the flow $z \mapsto e^{it}z$) would do.

(d) The mod-2 degree of the identity and antipodal map is the same, which therefore does not exclude the existence of non-vanishing vector fields on even-dimensional spheres.

3.3.19. The point of this exercise is that intersections of $Z \times Z$ with $\Delta \subset Y \times Y$ occur on the diagonal in $Z \times Z$, and even after a small perturbation of $Z \subset Y$ remain in a small neighborhood of this diagonal, which is orientable even when Z is not. In fact such a neighborhood is diffeomorphic to a neighborhood of the zero section in $\pi : TZ \to Z$. There a tangent space $T_v(TZ)$ has subspace $T_{\pi(z)}Z$ with the quotient space also equal to $T_{\pi(v)}Z$. Therefore the reversal of an orientation on $T_{\pi(v)}Z$ does not affect the orientation of $T_v(TZ)$ (again because $-$ times $-$ equals $+$).

Problem (*) A step of the Gram-Schmidt orthogonalization, which consists in normalizing a basis vector to the unit length and subtracting a multiple of it from subsequent basis vectors until they becomes orthogonal it, can be done *gradually*. This provides a deformational retraction of $GL_n(\mathbb{R})$ to O_n, and in particular identifies the connected components of the groups.

One way to show that SO_n (and hence $GL_n^+(\mathbb{R})$) is connected is based on studying the geometry of orthogonal transformation U. Its eigenvalues must have absolute value 1, and if not equal to ± 1, must come in pairs $e^{\pm i\theta}$ of complex conjugates. The corresponding eigenvectors can be taken in the form $u \pm iv$, where $u, v \in \mathbb{R}^n$ span a U-invariant plane on which U acts as a rotation through the angle θ. The orthogonal complement to this plane is U-invariant, and one can continue this geometric analysis on U by descending induction on n. When an eigenvalue is ± 1, the corresponding eigen-line can be taken real, whose orthogonal complement is still U-invariant (so our induction proceeds). Thus, in a suitable orthonormal basis the matrix of U is block diagonal with the blocks either ± 1 of size one, or size two of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Note that two blocks of size 1 both with the same eigenvalue ± 1 make one rotation block with $\theta = 0$ or π respectively. Consequently, in a suitable orthonormal basis the matrix of an orthogonal transformation with determinant $+1$ consists of the blocks of rotations through the angles $0 \leq \theta_i \leq \pi$ (and, for odd n, of one more block 1 of size one). Multiplying each θ_i by $t \in [0, 1]$, we connect U to I by a curve of orthogonal transformations.

Another way consists of fibering O_n over S^{n-1}, with each fiber actually diffeomorphic to O_{n-1}, by associating to an orthogonal transformation U the unit vector Ue_1 (where e_1 is the 1st vector of the standard basis in $\mathbb{R}^n$). As in 3.3.2(a), we can gradually rotate $\mathbb{R}^n$ about the orthogonal complement $\mathbb{R}^{n-2}$ to a/the plane containing e_1 and Ue_1 until Ue_1 is transformed to e_1. Calling the final rotation V, we thus obtain a curve in O_n connecting U with VU. The latter preserves e_1 and hence the orthogonal complement $Span(e_2, \ldots, e_n)$ on which VU still acts as a transformation from O_{n-1}. Thus O_n has no more components than O_{n-1} does, and this argument works as long as n remains greater than 1. Since $O_1 = \{\pm 1\}$ has two components so does O_n.