Answers to HW7

A. Given $\mathbb{R}^n \to \mathbb{R}^n : x \mapsto y = f(x)$ such that df_0 is invertible, introduce $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, (x, y) \mapsto z = f(x) - y$. It satisfies the hypothesis $\det(\partial z/\partial x)(0,0) \neq 0$ of the Implicit Function Theorem, and hence locally near $(x, y) = (0, 0)$ there exists $g : \mathbb{R}^n \to \mathbb{R}^n : y \mapsto g(x)$ such that $f(g(y)) = y$. To show that $g(f(x) = x$,we note that by the chain rule dg_0 is invertible, and hence by the same logic there exists a local map $\tilde{f}$ such that $g(\tilde{f}(x)) = x$. Applying f, we conclude that $\tilde{f}(x) = f(x)$.

Conversely, given $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, (x, y) \mapsto z = F(x, y)$ such that $\det(\partial z/\partial y)(0, 0) \neq 0$, introduce $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $(x, y) \mapsto (w, z) = (x, F(x, y))$. The Jacobi matrix of this map $\begin{bmatrix} I_n & 0 \\ F_x & F_y \end{bmatrix}$ at $(x, y) = (0, 0)$ satisfies the hypotheses $\det \neq 0$ of the Inverse Function Theorem. Therefore locally near $(w, z) = (0, 0)$ there exists the inverse map $(z, w) \mapsto (x, y) = (X(w, z), Y(w, z)$. In particular, at $z = 0$, we have $X(x, 0) = x$ and $F(x, Y(x, 0))) = 0$, i.e. $g(x) := Y(x, 0)$ is the function implicitly defined by $F(x, y) = 0$ and by $g(0) = 0$.

B. The velocity vector field $z \mapsto iz$ of the rotation flow in $\mathbf{C}^n$: $z \mapsto e^{it}z$ is tangent to the unit sphere $|z| = 1$, nowhere vanishes, and being centrally-symmetric, descends to a nowhere vanishing vector field on $\mathbb{R}P^{2n-1}$. as a nowhere. The vector field in $\mathbf{C}^n \times \mathbb{R}$ given by the (same) formula $(z, x) \mapsto (iz, 0)$ likewise descends to a vector field on $\mathbb{R}P^{2n}$ which has exactly one zero: on the axis $(0, x)$ of the rotation $(z, x) \mapsto (e^{it}z, x)$. In the chart $x = 1$ the vector field is defined by the invertible linear transformation $z \mapsto iz$. This assures transversality of the vector field as a section of the tangent bundle $T^{\mathbb{R}}P^{2n}$ to the zero section, and shows that the mod-2 self-intersection index of the zero section with itself equals 1. Since all vector fields on a manifold are homotopic to each other (since they form a vector space), any vector field on $\mathbb{R}P^{2n}$ must have zeroes (for otherwise the corresponding section would be transverse to the zero section and the have mod-2 intersection index with it equal 0).

C. In $\mathbb{R}^{m+n+1}$ transverse linear subspaces $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$ intersect along a 1-dimensional subspace. This translates into the fact that in $\mathbb{R}P^{m+n}$, the corresponding projective subspaces $\mathbb{R}P^m$ and $\mathbb{R}P^n$ (they consist of the 1-dimensional subspaces in those linear spaces) intersect transversally at one point. However, in S^{m+n} two compact submanifolds M^m and N^n of dimensions $m, n < m + n$ would have mod-2 intersection index equal 0. Indeed, the stereographic projection from a point p outside $M \cup N$ identifies $S^{m+n} - p$ with $\mathbb{R}^{m+n}$, where the compact submanifolds can be translated away from each other. The exception is the intersection index of the entire S^{m+n} with a point, which is equal to 1. Thus, when $m + n > 1$, $\mathbb{R}P^{m+n}$ contains M^m and N^n with non-zero intersection index, while in S^{m+n} such submanifolds don't exist. This implies that $\mathbb{R}P^k$ and S^k are not diffeomorphic when $k > 1$. For $k = 1$, they are diffeomorphic (and for $k = 0$ not).

2.4.3. For $X \subset \mathbb{R}^N$, the map $F : X \times \mathbb{R}^N \to \mathbb{R}^N : (x, a) \mapsto x + a$ is obviously a submersion and hence is transverse to $Y \subset \mathbb{R}^N$. Thus, $Z := F^{-1}(Y)$ is a submanifold in $X \times \mathbb{R}^N$. The regular values of its projection to $\mathbb{R}^N$ are exactly those $a \in \mathbb{R}^N$ for which the translate $X + a$ is transverse to Y. By Sard's lemma, such a form a full measure subset in $\mathbb{R}^N$.

2.4.13. (a, c) In $Y = \mathbb{R}$, take $Z = \{x > 0\}$, X a point x_0, and $W = \{x \leq x_0\}$. Then for $x_0 < 0$, we have $I_2(X, Z) = 0$ and for $x_0 > 0$, $I_2(X, Z) = 1$, despite that $X = \partial W$. (b,d) In $Y = \mathbb{R}^2$, two parallel lines X and Y can be deformed into crossing ones intersection at 1 point, although X is the boundary of the half-plane. (e) Take $W = \{x \leq 1\} \subset \mathbb{R} = Z = Y$.