Answers to HW6

2.1.10. I am not sure what your long pages of solutions are about. A manifold with boundary, in local coordinates $(x_1, ..., x_n)$ is the half-space $x_n \geq 0$. So, x_n is a function which is non-negative, vanishes exactly on the boundary, and has *negative* (rather than positive, as the book mistakenly asserts) derivative in (any) exterior direction.

2.2.2. The meaning of the exercise is to show that there may be no *interior* fixed point in Brouwer's theorem. As many of you noticed, a constant map from B^n to a point on its boundary gives a decisive example. Perhaps, a smarter map: $x \mapsto x/2$ from the ball of radius 1 centered at $(1, 0, ..., 0)$ also solves Exercise 2.1.4: If the ball is open, then there may be no fixed points at all.

2.2.5. If $f(0) > 0$ and $f(1) < 1$, then $f(x) = x$ at some $0 < x < 1$ by the Intermediate Value Theorem.

A. $SU_2 \times SU_2 \ni (g, h) \mapsto [\phi_{g,h} : x \mapsto gxh^{-1}]$, where $\phi_{g,h}$ is a linear transformation on the space $\mathbf{C}^2$ of matrices $x = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$ (of quaternions, if you wish) which preserves the determinant $|z|^2 + |w|^2$ of the matrix, and is therefore orthogonal in $\mathbf{C}^2 = \mathbb{R}^4$. Since $SU_2 \times SU_2$ is connected, the map lands in $SO(4)$ (rather than $O(4)$). The kernel of this homomorphism consists of all pairs (g, h) such that $gx = xh$, which for $x = 1$ yields $g = h$, while $gx = xg$ for all x implies $g = \pm 1$ (if you wish, the only quaternions commuting with $x = i, j, k$ are the reals). Thus, we have an injective smooth homomorphism $SU_2 \times SU_2 / \pm (1, 1) \to SO(4)$ of 6-dimensional Lie groups, whose differential at $(1, 1)$, therefore, induces an isomorphism of the Lie algebras. Since ϕ, as a Lie group homomorphism, has constant rank, its image is open in SO_4 — and closed, since $SU_2 \times SU_2$ is compact. Therefore the rang of ϕ is a connected component, i.e. the whole of SO_4 since, as it is not hard to check, say, by induction on n, SO_n is connected. We conclude that ϕ defines a surjective, and hence bijective Lie group isomorphism between $SU_2 \times SU_2 / \pm (1, 1)$ and SO_4.

B. The Lie algebra of vector fields $\alpha d/dx + 2\beta x d/dx + \gamma x^2 d/dx$ on the line generates, as we know from the previous hw, the group of *Möbius transformations* $x \mapsto (ax + b)/(cx + d)$, whose Lie algebra can be also identified with the space of real traceless matrices $\begin{bmatrix} -\beta & \alpha \\ \gamma & \beta \end{bmatrix}$ with the commutator operation. On the other hand, the cross-product Lie algebra is isomorphic to that of SO_3, and hence of SU_2. The latter consists of traceless anti-Hermitian matrices $\begin{bmatrix} i\omega & -\mu + i\nu \\ \mu - i\nu & -i\omega \end{bmatrix}$, with the Lie operation being the commutator of matrices again. Clearly, over complex numbers (i.e. assuming that α, β, γ in the first case, and ω, μ, ν in the second are complex), both spaces coincide with the Lie algebra of complex traceless 2×2-matrices with respect to the commutator operation.