Answers to HW5

1. It's straightforward:

$$\sum v_i\partial_i \sum w_j\partial_j - \sum w_i\partial_i \sum v_j\partial_j = \sum_j \left(\sum_i v_i\partial_i w_j - w_i\partial_j v_j \right) \partial_j.$$

2. When $v_i = \sum_k A_{ik}x_k$ and $w_j = \sum_k B_{jk}x_k$, for the jth component of $[v_A, v_B]$, we find:

$$\sum_i \sum_k (A_{ik}x_k B_{ji} - B_{ik}x_k A_{ji}) = \sum_k \left(\sum_i B_{ji}A_{ik} - A_{ji}B_{ik} \right) x_k,$$

wich is the linear vector field with the coeficient matrix $BA - AB$ (and differs by the sign from what I predicted).

3. It is also straightforward. Maybe it is more interesting to directly check the following: Given a vector $\mathbf{u}$ in $\mathbb{R}^3$, the cross product operation: $\mathbf{r} mapsto \mathbf{u} \times \mathbf{r}$ (where $\mathbf{r} = (x, y, z)$ is a radius-vector), defined a linear map whose matrix is anti-symmetric (and any anti-symmetrix matrix is obtained this way). Then, in view of Exercise 2, the current problem reduces to the Jacobi identity for the cross-product:

$$\mathbf{v} \times (\mathbf{w} \times \mathbf{r}) - \mathbf{w} \times (\mathbf{v} \times \mathbf{r}) = (\mathbf{v} \times \mathbf{w}) \times \mathbf{r}.$$

The latter identity is easy to verify on the triples formed from the basis unit vectors i, j, k (satisfying $i \times j = k$).

4. Indeed, $[\partial_x, x\partial_x] = p_x$, $[\partial_x, x^2\partial_x] = 2x\partial_x$, and $[x\partial_x, x^2\partial_x] = x^2\partial_x$. Linear transfromations on the plane $\mathbb{R}^2$ act on the real projective line $\mathbb{R}P^1$ (whose points are 1-dimensional subsaces in $\mathbb{R}^2$) by fractional-linear transformations $x \mapsto (ax + b)/(cx + d)$ (where x is the slope of an 1-dimensional subspace). The Lie algebra of this 3-dimenasional transformation group is realized by the vector fields $(\alpha + \beta x + \gamma x^2)\partial_x$.

5. The flow of the vector field $\dot{x} = Ax$ is $x \mapsto y = e^{tA}x$. Its differential $e^{tA} : T_x \to T_y$ acts on the vector Bx of the vector field v_B yielding $e^{tA}Bx \in T_y$. Substituting $x = e^{-tA}y$, we find the transformed vector field as $y \mapsto e^{tA}Be^{-tA}y$. Differentiating in t at $t = 0$, we find $y \mapsto (AB - BA)y$. Up to a sign, this is the linear vector filed $[v_A, v_B]$. Th point is that the same holds for non-linear vector fields: the velocity with which the flow of a vector field v tranforms a vector filed w is (up to a sign, which is a matter of some convention) the Lie bracket $[v, w]$.