Answers to HW4

Problem 8, Section 8. As the hint says, move each point $x \in S^n$ in the direction of $v(x)/|v(x)|$ along the great circle of the sphere with the speed π radians per minute to reach $-x$ after 1 minute.

Problem 9, Section 8. The function $TX \to \mathbb{R}$ taking a tangent vector $v \in T_x X$ to its square length $|v|^2$ (it is assumed that X is embedded into a Euclidean space, and the lengths of tangent vectors are induced by the respective embedding of the tangent spaces). It is a smooth function on T_X whose critical points are $v = 0$ (zero tangent vectors). Thus any non-zero value of this function is regular, and hence the level set $\{v \in TX \mid |v|^2 = 1\}$ is a smooth hypersurface in TX.

Problem 12, Section 8. The differential of $(x, y) \mapsto (u, v, w) = (x, xy, y^2)$ is given by the Jacobi matrix

$$\begin{bmatrix} 1 & 0 \\ y & x \\ 0 & 2y \end{bmatrix}$$

whose 2×2-minors are equal to x, $2y$, and $2y^2$. They all vanish only at $x = y = 0$, and hence everywhere else the map is an immersion. The range satisfies the equation $v^2 = u^2 w$. It can be visualized by drawing sections $w = const$, which for $w > 0$ are two crossing lines with angle between them getting smaller as w decreases, so that at $w = 0$ the two lines merge. Interestingly, when $w < 0$, the ray $u = v = 0$ still satisfies the equation although it does not belong to the range of our map $\mathbb{R}^2 \to \mathbb{R}^3$. What happens is that these real values of (u, v, w) correspond to imaginary values of $(x, y) = (0, \pm i\sqrt{|w|})$. With this ray included, the surface can be visualized as a funny umbrella which doesn't quite save from rain (see Figure).

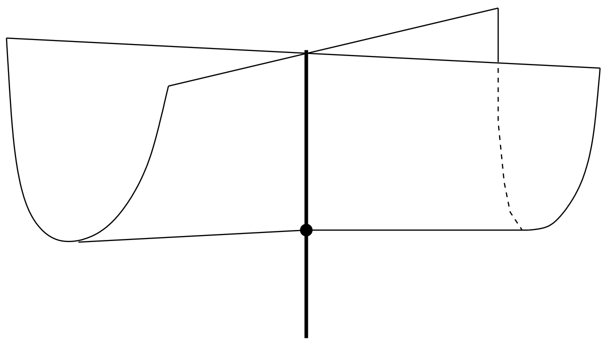

FIGURE 1. Whitney's umbrella

Problem A. By completing squares, make change of variables depending on parameters:

$$\begin{aligned} x^2y + y^2 - ax - by &= (y + \frac{x^2}{2})^2 - b(y + \frac{x^2}{2}) - \frac{x^4}{4} + b\frac{x^2}{2} - ax \\ &= (y + \frac{x^2}{2} - \frac{b}{2})^2 - \frac{b^2}{4} - \frac{x^4}{4} + b\frac{x^2}{2} - ax \\ &= -\frac{x^4}{4} + b\frac{x^2}{2} - ax - \frac{b^2}{4} + z^2, \end{aligned}$$

where $z := y + x^2/2 - b/2$ together with x form a new coordinate system. The advantage is that now the problem is reduced to studying critical points of one variable functions: $p(x) := -x^4/4 + bx^2/2 - ax$. Those with degenerate critical points are found from $p'(x) = p''(x) = 0$, which yield the parametric curve $a = 2x^3$,$b = 3x^2$ on the (a, b)-plane. This curve (known as the *semicubical parabola* $(a/2)^2 = (b/3)^3$, or the *cusp*) divides the plane into two connected regions. Inside the cusp, polynomials p have 2 local maxima and 1 local minimum, and outside the cusp, 1 local maximum only. THis is easily found by sketching the graphs of p with $(a, b) = (0, \pm 1)$. Respectively the functions of the original family (equivalent to $p(x) + z^2$) have 2 saddles and one local minimum in the former case, and just one saddle in the latter.

Problem B. To follow the Lagrange multiplier method for finding constrained extrema, we introduce:

$$F(x, y, z, \lambda) := ax^2 + by^2 + cz^2 - \lambda(x^2 + y^2 + z^2 - 1),$$

an auxiliary function in four variables. Its critical points are found from $F_x = F_y = F_z = F_\lambda = 0$ which yield:

$$(ax, by, cz) = \lambda(x, y, z) \text{ together with } x^2 + y^2 + z^2 = 1.$$

Since a, b, c are distinct, solutions are $(\pm 1, 0, 0)$ with $\lambda = a$, $(0, \pm 1, 0)$ with $\lambda = b$, and $(0, 0, \pm 1)$ with $\lambda = c$. They correspond to the critical points of our function on the sphere (obtained by simply dropping the value of λ). The quadratic differentials of the function at the critical points are easily computed. For example, near $\pm(1, 0, 0)$, (y, z) form a chart on the sphere with $x = \pm\sqrt{1 - y^2 - z^2}$ and hence $ax^2 + by^2 + cz^2 = a + (b - a)y^2 + (c - a)z^2$. This is a non-degenerate quadratic function with a local maximum at $(y, z) = (0, 0)$ under our assumption $a > b > c$. Likewise, $(0, \pm 1, 0)$ turn out to be saddles, and $(0, 0, \pm 1)$ non-degenerate local minima.