Answers to HW2

1. (a) For $g(u,v) := f(x(u,v), y(u,v))$, by the chain rule we have $f_x dx + f_y dy = (f_x x_u + f_y y_u)du + (f_x x_v + f_y y_v)dv = g_u du + g_v dv$ for any smooth substitution $x = x(u,v), y = y(u,v)$.

(b) For $x = u+v, y = v$ (using for shortness $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial_y$ for shortness), we have: $\partial_u = x_u\partial_x + y_u\partial_y = \partial_x$, $\partial_v = x_v\partial_x + y_v\partial_y = \partial_x + \partial_y$, hence $g_u = f_x$, $g_v = f_x + f_y$, and therefore for the gradient vector fields

$$g_u\partial_u + g_v\partial_v = f_x\partial_x + (f_x+f_y)(\partial_x+\partial_y) = (2f_x+f_y)\partial_x + (f_x+f_y)\partial_y \neq f_x\partial_x + f_y\partial_y.$$

2. $\partial_\theta = x_\theta\partial_x + y_\theta\partial_y = (-r\sin\theta)\partial_x + (r\cos\theta)\partial_y = -y\partial_x + x\partial_y$.

3. The map $(t,\theta) \mapsto (e^t\cos\theta, e^t\sin th)$ will do. It parameterizes $\mathbb{R}^2 - 0$ by polar coordinates combined with the change $r = e^t$.

4. The differential of $t \mapsto (x,y,z) = (t^2, t^3, t^4)$ is $(dx, dy, dz) = (2t, 3t^2, 4t^3)dt$, which has rank 0 only at $t = 0$. The range of the map satisfies the equations $y^2 = x^3$ and $z = x^2$. Thus, it can be pictured as the graph of the 2-valued function $y = \pm x^{3/2}$ (which — the graph — is called the semicubical parabola, and has the shape of a curve with the cusp we've seen in the last lecture), lifted from the (x,y)-plane to the surface of the paraboloid $z = x^2$ in space. The functions $f(x,y,z) = y^2 - x^3$ and $g(x,y,z) = z - x^2$ define a map $\mathbb{R}^3 \to \mathbb{R}^2$ with the differential $df = (-3x^2)dx + 2ydy + 0dz$, $dg = (-2x)dx + 0dy + 1dz$. The rank of it drops from 2 to 1 only on the line $x = y = 0$. which intersects our curve only at the origin $x = y = z = 0$. Thus, this point excluded, the rest of the curve is parameterized by an immersion and is also a fiber of a submersion, hence a submanifold in $\mathbb{R}^3$.

5. It was done in class for $n = 3$, but in the notation which serves all n. The group $O(n)$ is the inverse image of the identity I under the map $f : U \mapsto U^tU$ from the space $\mathbb{R}^{n^2}$ of all $n\times n$-matrices to the space $\mathbb{R}^{n(n+1)/2}$ of symmetric $n \times n$-matrices. Its linearization $df_I : T_I\mathbb{R}^{n^2} \to T_I\mathbb{R}^{n(n+1)/2}$ at $U = I$ is given by

$$A \mapsto \frac{d}{d\epsilon}|_{\epsilon=0}(I+\epsilon A)^t(I+\epsilon A) = A^t + A.$$

The tangent space $T_IO(n)$ is the null-space of df_I, and is therefore the space of matrices A satisfying $A^t = -A$.