Answers to HW13

1. Using Cartan's homotopy formula $L_v = di_v + i_v d$, we have:

$$(g^t)^*\omega - \omega = \int_0^t \frac{d}{d\tau}(g^\tau)^*\omega \; d\tau = \int_0^t L_v (g^\tau)^*\omega \; d\tau = d\int_0^t i_v (g^\tau)^*\omega \; d\tau,$$

when ω (and hence $(g^\tau)^*\omega$) is closed, and so

$$\int_0^1 (g^t)^*\omega \; dt = \omega + d\int_0^1 dt \int_0^t i_v (g^\tau)^*\omega \; d\tau.$$

2. Let $\hat{\omega}$ stands for the average of a differential form ω on the torus $T^n = (\mathbb{R}/\mathbb{Z})^n$ over its translations $h^*\omega$ by the elements h of the torus with respect to the translation-invariant measure. By Problem 1, when ω is closed, $\hat{\omega}$ represents the same cohomology class. Conversely, if a translation-invariant form $\hat{\omega}$ is exact, i.e. $\hat{\omega} = d\alpha$, then by taking the average we find that $\hat{\omega} = d\hat{\alpha}$, i.e. it is exact already in the complex of translation-invariant forms. In fact, the translation-invariant forms on the torus have constant coefficients in $\mathbb{R}^n$, i.e. they form the complex $\Lambda^\bullet \mathbb{R}^{n*}$ of exterior forms with the zero De Rham differential. Thus, $H^\bullet_{DR}(T^n) = \Lambda^\bullet \mathbb{R}^{n*}$.

3. The answer is that every closed compactly supported differential form ω in $\mathbb{R}^n$ of degree $k < n$ is the differential of a compactly supported $k-1$-form, while the cohomology class of a compactly supported n-form is uniquely determined by the integral $\int_{\mathbb{R}^n} \omega$. The case $k = 0$ is obvious. For $k > 0$, let ω^k is a compactly supported k-form in $\mathbb{R}^n$, such that $\int_{|R^n} \omega = 0$ for $k = n$. It suffices to prove that ω is the differential of a compactly supported $k-1$-form.

By the Poincaré lemma $\omega^k = d\psi^{k-1}$ where however ψ does nor have to be compactly supported. However, outside a ball B containing the support of ω, the $k-1$-form ψ is closed. When $k = 1$, ψ is a constant function outside B, and subtracting this constant, we obtain a compactly-supported function whose differential is ω^1. To similarly correct ψ^{k-1} when $k > 1$, we note that the exterior $\mathbb{R}^n - B$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$, the 1-dimensional vector bundle over S^{n-1}. Moreover, by Stokes' formula, $\int_B \omega = \int_{\partial B} \psi = 0$. By the "bundle" version of the Poincareé lemma $\mathbb{R}^n - B$ has the De Rham cohomology of S^{n-1}. So, the problem reduces to the fact that for $k > 1$, a closed $k-1$-form on S^{n-1} which in the case when $k = n$ integrates to 0 over the sphere, is exact. Indeed, then $\psi^{k-1} = d\alpha^{k-2}$ in $\mathbb{R}^n - B$. Multiplying α by a smooth function equal to 1 outside $2B$ and 0 on B, we extend it as $\tilde{\alpha}$ to the whole of $\mathbb{R}^n$ and find that $\omega = d(\psi - d\tilde{\alpha})$, where $\psi - \tilde{\alpha} = 0$ outside $2B$.

Actually in order to prove the description we used for the De Rham cohomology of the spheres S^{n-1}, one needs to proceed inductively on n with a similar argument. Namely, the sphere S^n can be glued by means of two stereographic projections from two charts $\mathbb{R}^n\pm$ intersecting over $S^{n-1}\times\mathbb{R}$. A closed k-form ω on S^n can be written as $\omega = d\psi_\pm^{k-1}$ on the charts by means of the Poincaré lemma, where $\psi_+ - \psi_-$ is closed on the intersection $S^{n-1}\times\mathbb{R}$, and hence (by the induction hypothesis) exact under the assumption that $\int_{S^{n-1}}\psi_+ - \int_{S^{n-1}}\psi_- = \int_{S^n}\omega = 0$. This allows to correct ψ_- by the differential of a $k-2$-form on $S^{n-1}\times\mathbb{R}$ so that the result matches ψ_+ and yields a globally defined $k-1$-form ψ on S^n such that $d\psi = \omega$. This establishes the theorem: $H^k_{DR}(S^n) = \mathbb{R}$ for $k = 0, n$, and $= 0$ otherwise.

4. In a connected manifold X, any two points p and p' have coordinate neighborhoods B, B' which can be isotoped into each other by a family of diffeomorphisms $g_t : X \to X$, i.e. $g_0 = id_X$, and $g_1(B) = B'$. Therefore a top-degree form ω' supported in B' represents the same class in $H^{top}_{DR}(X)$ as $g_1^*\omega'$ supported in B. By Problem 3, the cohomology class of a top-degree form in the compactly-supported De Rham complex of B is determined by the integral of the form over B (and hence over the entire X).

Now, let ω be an n-form on a closed oriented connected n-dimensional manifold X. Using partition of unity $\sum \rho_i = 1$ subordinate to an atlas of coordinate balls B_i such that $g_i(B) = B_i$ for some diffeomorphisms $g_i : X \to X$ homotopic to the identity through diffeomorphisms. Write $\omega = \sum_i \rho_i\omega_i$ to conclude that it is cohomologous to $\sum_i g_i * \rho_i\omega$ supported in B. Thus the cohomology class of ω is uniquely determined by $\int_B \sum_i g_i^*\rho_i\omega_i = \int_X \omega$.

5. Let a surface in $\mathbb{R}^3$ be the graph $z = f(x, y)$ of a smooth function with the critical point at the origin $(x, y) = (0, 0)$. By the orthogonal diagonalization theorem, we can rotate the coordinate system on the (x, y)-plane so that the quadratic differential $d^2f/2 = f_{xx}(0,0)(dx)^2/2 + f_{xy}(0,0)dxdy + f_{yy}(0,0)(dy)^2/2$ becomes (in the rotated coordinates) $k_1(dx)^2/2 + k_2(dy)^2/2$. Here k_1, k_2 are the eigenvalues of the pair of quadratic forms d^2f and $(dx)^2 + (dy)^2$, and so the product k_1k_2 is equal to the determinant $\det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}(0,0)$. In the rotated coordinate, we have $z = k_1x^2/2 + k_2y^2/2 + o(x^2 + y^2)$. Let us compute the Gauss map near $(0, 0)$. The tangent planes have the equations $dz = (k_1x + ...)dx + (k_2y + ...)dy$, where $... = o(\sqrt{(x^2 + y^2)})$. A normal vector has the components $(-k_1x + ..., -k_2y + ..., 1)$, an the length of the form $\sqrt{1+o(x^2+y^2)}$. Thus the linear approximation to the Gauss map at the origin, which is the linear map from the tangent plane to the surface at $0,0,0)$ to the tangent plane to the unit sphere at the point $(0,0,1)$ has the form $(dx,dy)\mapsto(-k_1dx,-k_2dy)$. The Jacobian determinant therefore equals k_1k_2, which coincides with the Hessian determinant.