Answers to HW12

1. By Stokes' formula, $\int_c d\alpha = \int_{\partial c} \alpha = 0$ since $\partial c = 0$, and likewise $\int_{\partial c} \omega = \int_c d\omega = 0$ since $d\omega = 0$.

2. $d(\psi \wedge \beta) = (d\psi) \wedge \beta + (-1)^{\bar{\psi}} \psi \wedge d\beta = \alpha \wedge \beta$ when $\alpha = d\psi$ and $d\beta = 0$.

3. The flow of v to preserve ω is equivalent to $L_v\omega = 0$. By Cartan's homotopy formula and in the case at hands where $d\omega = 0$, this means $di_v\omega = 0$. By Poincare's Lemma, this is equivalent to $i_v\omega = -dH$, where H (in our case) is a function. For $\omega = \sum_i dp_i \wedge dq_i$, and $dH := \sum_i (H_{p_i} dp_i + H_{q_i} dq_i)$ the relation $i_v\omega = -dH$ translates into $v = \sum_i (H_{p_i} \partial_{q_i} - H_{q_i} \partial_{p_i})$.

4. Since $V/|x|^n$ is invariants under $x \mapsto e^t x$, we have $L_E V/|x|^n = 0$, and since $dV = 0$, conclude from Cartan's homotopy formula that $i_E V/|x|^n$ is closed. By Stokes' formula, the integral over the boundary of the cube is equal to the integral over the boundary of a small (and hence any) ball centered at the origin. On the boundary $|x| = 1$ of the unit ball B, the integral reduces to $\int_{\partial B} i_E V = \int_B di_E V = n \int_B V$, i.e. n times the volume of the unit n-dimensional ball (i.e. the same as the $n-1$-dimensional "surface area" of the unit sphere).

5. By the fundamental theorem of calculus, a 2π-periodic 1-form $\phi(x)dx$ on $\mathbb{R}$ is the differential df of 2π-periodic function if and only if $\oint \phi(x)dx = 0$. Therefore integration over the circle establishes an isomophism $H^1_{DR}(S^1) = \mathbb{R}$. We also have $H^0_{DR}(S^1) = \{\text{constant functions on } S^1\} = \mathbb{R}$.