Answers to HW11

1. Given a non-vanishing differential n-form ω on an n-dimensional manifold X, the following rule defines an orientation on X: A basis $v_1, \dots, v_n$ in T_xX is right-oriented iff $\omega_x(v_1, \dots, v_n) > 0$. (In a local chart, $\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n$, where ϕ is a non-vanishing smooth function. Therefore, if $\omega_x(\partial_1, \dots, \partial_n) > 0$ at $x = 0$, it will be true for any other point x within the connected component of the chart.)

Conversely, given an orientation on X, take a partition of unity $1 = \sum_i \rho_i$ subordinate to an atlas of coordinate charts on X. For each i, let $\omega_i = dx_1 \wedge \ldots \wedge dx_n$ be the right-oriented volume form in a chart containing the support of ρ_i. Then $\omega := \sum_i \rho_i \omega_i$ (where each summand is extended by zero to a global n-form on X) is a nowhere vanishing n-form on X. Indeed, on each T_xX it is the sum of exterior n-forms which are all non-negative proportional to each other, and of which at least one is non-zero (since $\sum_i \rho_i(x) = 1$).

2. We have $dA = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$, and consequently $A \wedge (dA)^{\wedge n} = (-1)^n n! du \wedge dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n$, which is a constant-coefficient non-zero top degree form in $\mathbb{R}^{2n+1}$.

3. In local coordinates, $F^1 = \sum_i \phi_i(x) dx_i$, and $dF = 0$ is equivalent to the "Clairaut constraints": $\partial_j \phi_i = \partial_i \phi_j$ for all i, i. These coincides with $L_u F(v) - L_v F(u) = F([u, v])$ for $u = \partial_j$ and $v = \partial_i$ since $[\partial_j, \partial_i] = 0$. One way to prove the converse is to notice that $G(u, v) := L_u F(v) - L_v F(u) - F([u, v])$ is anti-symmetric in u, v and $C^\infty(X)$-linear in each of them. Thus G is a differential 2-form $\sum_{i<j} g_{ij}(x) dx_i \wedge dx_j$, where $g_{ij} = G(\partial_j, \partial_i) = \partial_j \phi_i - \partial_i \phi_j$.

4. On the one hand, $dF = (Q_x - P_y)\ dx \wedge dy =$

$$(Q_x(x(u,v), y(u,v)) - P_y(x(u,v), y(u,v))) \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} du \wedge dv.$$

On the other, $P dx + Q dy = \mathcal{P} du + \mathcal{Q} dv$, where

$$\mathcal{P} = P(x(u,v), y(u,v)) x_u + Q(x(u,v), y(u,v)) y_u,$$
$$\mathcal{Q} = P(x(u,v), y(u,v)) x_v + Q(x(u,v), y(u,v)) y_v.$$

Therefore $dF = (\mathcal{Q}_u - \mathcal{P}_v)\ du \wedge dv$, where

$$\begin{aligned} \mathcal{Q}_u - \mathcal{P}_v &= (P_x x_u + P_y y_u) x_v + (Q_x x_u + Q_y y_u) y_v \\ &\quad - (P_x x_v + P_y y_v) x_u - (Q_x x_v + Q_y y_v) y_u \end{aligned}$$

$$= Q_x\ (x_u y_v - x_v y_u) - P_y\ (y_v x_u - y_u x_v) = (Q_x - P_y)\ \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

5. The flux is usually defied as $\int_D (w \cdot n) dA$, where the "exterior" normal vector $n = (r_u \times r_v)/|r_u \times r_v|$ (in terms of the radius-vector $r = (x, y, z)$), and the area element $dA = |r_u \times r_v|\ dudv$. Combining, we find the flux equal to

$$\int_D w \cdot (r_u \times r_v)\ dudv =$$

$$= \int_D \left(A \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} + B \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} + C \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right) dudv$$

$$= \int_{r:D \to \mathbb{R}^3} (A\ dy \wedge dz + B\ dz \wedge dx + C\ dx \wedge dy)$$

$$= \int_{r:D \to \mathbb{R}^3} i_{A\partial_x + B\partial_y + C\partial_z}\ dx \wedge dy \wedge dz.$$