Answers to HW10

1. For a fixed $k \times n$-matrix A, consider $\det AB$ as a k-linear totally anti-symmetric function F (i.e. as an exterior k-form) of the columns of the $n \times k$-matrix B. Just as any such function, it is a linear combination $F = \sum_I F_I \det B_I$ with the coefficients $F_I = F(e_{i_1}, \dots, e_{i_k}) = \det A_I$.

2. Consider $\det A$ with fixed last $n-k$ rows of the $n \times n$-matrix A as an k-linear totally anti-symmetric function of the first k rows. Just as any such function it is a linear combination of $\det A_I$, $I = \{i_1 < \dots < i_k\}$, with coefficients c_I which are the values of this function when the first k rows are (transposed) unit coordinate vectors $e_{1_1}, \dots, e_{i_k}$. Performing $(i_1 - 1) + (i_2 - 1) + \dots + (i_k - 1)$ transpositions, move the columns $i_1, \dots, i_k$ to the positions $1, \dots, k$. Then the matrix assumes the block-triangular form $\begin{bmatrix} I_k & 0 \\ * & A'_I \end{bmatrix}$ with the determinant equal to $\det A'_I$. Thus $c_I = (-1)^{|I|} \det A'_I$.

`Remark:` Another proof: $x_1 \wedge \cdots \wedge x_n = (x_1 \wedge \cdots x_k) \wedge (x_{k+1} \wedge \cdots \wedge x_n)$.

3. Let α be a linear form taking value 1 on the vector v. Then for any exterior k-form ω, we have $i_v(\alpha \wedge \omega) = \omega - \alpha \wedge (i_v \omega)$. Therefore, when $i_v \omega = 0$, we have $\omega = i_v(\alpha \wedge \omega)$.

4.2.2. WLOG, assume that $\phi_n = \sum_{i=1}^n c_i \phi_i$. Then

$$\phi_1 \wedge \cdots \wedge \phi_n = \sum_{i=1}^n c_i \phi_1 \wedge \cdots \wedge \phi_{n-1} \wedge \phi_i = 0$$

since the ith summand contains the wedge product $\phi_i \wedge \phi_i$.

4.2.6. (a) If $v'_i = \sum_j a_{ij} v_j$, then $T(v'_1, \dots, v'_k) = \det[a_{ij}] T(v_1, \dots, v_k)$, i.e. the transition matrix between the bases has positive determinant whenever $T(v1, \dots, v_k)$ and $T(v'_1, \dots, v'_k)$ have the same sign.

(b) When V^k is oriented, declare a basis element T in the 1-dimensional space $\Lambda^k V^*$ right-oriented if $T(v_1, \dots, v_k) > 0$ for right-oriented bases $(v_1, \dots, v_k)$. Obviously the same condition holds for any positive multiple of T (instead of T), and by (a) for any right-oriented basis when it holds for one of them.

(c) Conversely, given one of positive-proportional non-zero elements $T \in \Lambda^k V^*$, call a basis $(v_1, \dots, v_k)$ of V right-oriented if $T(v_1, \dots, v_n) > 0$. By (a) the condition holds for all same-oriented bases when it holds for one of them.