morphisms $\{\bar{\kappa}_n\}$ constitutes a direct sequence of groups in the usual sense (see, for example, [10, ch. VIII, definition VIII 12]). The limit group of this direct sequence of groups will be called the *limit group* of the given Leray-Koszul sequence.

In most cases we shall have to deal with Leray-Koszul sequences (A^n, d^n) for which each of the groups A^n is bigraded, $A^n = \sum A^n_{p,q}$, each of the differential operators d^n is homogeneous, and A^{n+1} inherits its bigraded structure from A^n. In this case each of the homomorphisms κ_n is homogeneous of degree (0, 0), and there is determined a bigraded structure on the limit group in a natural way. Also, it will usually be true that for each pair of integers (p, q) there exists an integer N such that if $n > N$, then κ_n maps $A^n_{p,q}$ isomorphically onto $A^{n+1}_{p,q}$. This makes it possible to determine any homogeneous component of the limit group by an essentially finite process.

4. Definition of an Exact Couple; The Derived Couple

An *exact couple* of abelian groups consists of a pair of abelian groups, A and C, and three homomorphisms:

$$f: A \to A,$$

$$g: A \to C,$$

$$h: C \to A.$$

These homomorphisms are required to satisfy the following "exactness" conditions:

$$\text{image } f = \text{kernel } g,$$

$$\text{image } g = \text{kernel } h,$$

$$\text{image } h = \text{kernel } f.$$

These three conditions can be easily remembered if one makes the following triangular diagram,

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd}$$

and observes that the kernel of each homomorphism is required to be the image of the preceding homomorphism. We shall denote such an exact couple by the notation $\langle A, C; f, g, h\rangle$. When there is no danger of confusion, we shall often abbreviate this to $\langle A, C\rangle$.

There is an important operation which assigns to an exact couple $\langle A, C; f, g, h\rangle$ another exact couple, $\langle A', C'; f', g', h'\rangle$, called the *derived* exact couple. This derived exact couple is defined as follows.

Define an endomorphism $d:C \to C$ by $d = g \circ h$. Then $d^2 = d \circ d = g \circ h \circ g \circ h = 0$, since $h \circ g = 0$ by exactness. Therefore d is a differential operator. Let $C' = \mathfrak{H}(C)$, the derived group of the differential group (C, d). Let $A' = f(A) =$ image $f =$ kernel g. Define $f':A' \to A'$ by $f' = f \mid A'$, the restriction of f to the subgroup A'. The homomorphism $h':C' \to A'$ is induced by h: it is readily verified that $h[\mathfrak{Z}(C)] \subset A'$, and $h[\mathfrak{B}(C)] = 0$, hence h induces a homomorphism of the factor group $C' = \mathfrak{Z}(C)/\mathfrak{B}(C)$ into A'. The definition of $g':A' \to C'$ is more complicated. Let $a \in A'$; choose an element $b \in A$ such that $f(b) = a$. Then $g(b) \in \mathfrak{Z}(C)$, and $g'(a)$ is defined to be the coset of $g(b)$ modulo $\mathfrak{B}(C)$. It is easily verified that this definition is independent of the choice made of the element $b \in A$, and that g' is actually a homomorphism.

Of course, it is necessary to verify that the homomorphisms f', g', and h' satisfy the exactness condition of an exact couple. This verification is straightforward, and is left to the reader.

It is clear that this process of derivation can be applied to the derived exact couple $\langle A', C'; f', g', h'\rangle$ to obtain another exact couple $\langle A'', C''; f'', g'', h''\rangle$, called the *second derived couple*, and so on. In general, we shall denote the n^{th} derived couple by $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)}\rangle$.

5. Maps of Exact Couples

Let $\langle A, C; f, g, h\rangle$ and $\langle A_0, C_0; f_0, g_0, h_0\rangle$ be two exact couples; a *map*,

$$(\phi, \psi): \langle A, C; f, g, h\rangle \to \langle A_0, C_0; f_0, g_0, h_0\rangle.$$

consists of a pair of homomorphisms,

$$\phi : A \to A_0,$$

$$\psi : C \to C_0,$$

which satisfy the following three commutativity conditions:

$$\begin{aligned} \phi \circ f &= f_0 \circ \phi \\ \psi \circ g &= g_0 \circ \phi \\ \phi \circ h &= h_0 \circ \psi. \end{aligned}$$

If $d = g \circ h:C \to C$ and $d_0 = g_0 \circ h_0:C_0 \to C_0$ denote the differential operators on C and C_0 respectively, then our definitions imply the following commutativity relation:

$$\psi \circ d = d_0 \circ \psi.$$

Therefore ψ is an allowable homomorphism, in the sense defined in the preceding section, and hence induces a homomorphism

$$\psi':C' \to C_0'$$

of the corresponding derived groups. Also, it is clear that $\phi(A') \subset A_0'$; therefore ϕ defines a homomorphism

$$\phi':A' \to A_0'.$$

It can now be verified without difficulty that the pair of homomorphisms (ϕ', ψ') constitute a map of the first derived exact couples,

$$(\phi', \psi'):\langle A', C'\rangle \to \langle A_0', C_0'\rangle$$

in the sense just defined. We will say that the map (ϕ', ψ') is *induced* by (ϕ, ψ). By iterating this process, one obtains a map $(\phi^{(n)}, \psi^{(n)}):\langle A^{(n)}, C^{(n)}\rangle \to \langle A_0^{(n)}, C_0^{(n)}\rangle$ which is induced by the given map (ϕ, ψ).

The set of all exact couples and maps of exact couples constitutes a category in the sense of Eilenberg and MacLane [3], and the operation of derivation is a covariant functor.

Let (ϕ_0, ψ_0) and $(\phi_1, \psi_1):\langle A, C; f, g, h\rangle \to \langle A_0, C_0; f_0, g_0, h_0\rangle$ be two maps of exact couples in the sense we have just defined. The maps (ϕ_0, ψ_0) and (ϕ_1, ψ_1) are said to be *algebraically homotopic*[2] (notation: $(\phi_0, \psi_0) \simeq (\phi_1, \psi_1)$) if there exists a homomorphism $\xi: C \to C_0$ such that for any element $c \in C$,

$$\psi_1(c) - \psi_0(c) = \xi[d(c)] + d_0[\xi(c)],$$

and for any $a \in A$,

$$\phi_1(a) - \phi_0(a) = h_0 \xi g(a).$$

It is readily verified that the relation so defined is reflexive, transitive, and symmetric, and hence is an equivalence relation. The main reason for the importance of this concept is the following proposition:

THEROEM 5.1. *If the maps*

$$(\phi_0, \psi_0), (\phi_1, \psi_1):\langle A, C; f, g, h\rangle \to \langle A_0 C_0; f_0, g_0, h_0\rangle$$

are algebraically homotopic, then the induced maps (ϕ_0', ψ_0') *and* (ϕ_1', ψ_1') *of the derived couples are the same.*

The proof is entirely trivial. It follows that the induced maps of the n^{th} derived couples, $(\phi_0^{(n)}, \psi_0^{(n)})$ and $(\phi_1^{(n)}, \psi_1^{(n)})$ are also the same.

6. Bigraded Exact Couples; The Associated Leray-Koszul Sequence

In the applications later on it will usually be true that groups occurring in the exact couples with which we are concerned will be bigraded groups, and that all the homomorphisms involved will be homogeneous homomorphisms. Then the groups of the successive derived couples will inherit a bigraded structure from the original groups, and the homomorphisms in the successive derived couples will also be homogeneous. To be precise, if $\langle A, C; f, g, h\rangle$ is a bigraded exact couple, and $\langle A', C'; f', g', h'\rangle$ denotes the first derived couple, then f' and f have the same degree of homogeneity, as do h' and h; however, the degree of homogeneity of g' is that of g minus that of f.

Let $\langle A, C; f, g, h\rangle$ be an exact couple, and let $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)}\rangle$, $n = 1, 2, \cdots$ denote the successive derived couples. Let $d^{(n)} = g^{(n)} \circ h^{(n)}: C^{(n)} \to C^{(n)}$ denote the differential operator of $C^{(n)}$. Then the sequence of differential groups

[2] This definition is patterned after a similar one given by J. H. C. Whitehead, [20].