TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 9

A. GIVENTAL

Recall that in the previous lectures we encoded the Gromov-Witten invariants $(t,\dots,t)_{0,n,d}$ into a generating function, the *genus* 0 *potential function*:

$$F(t,q)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{d}q^{d}(t,\dots,t)_{0,n,d},$$

and we used the potential function to define a new product structure on $H^*(X)$.

First let us recall some notation. Let $\{\phi_\alpha\}$ be a basis for $H^*(X)$. When we need to refer specifically to elements of this basis in H^0 and H^2 we will use the notation $\mathbf{1}$ and $p_1,\dots,p_r$ respectively.[1] In this notation we write the variable t as

$$t=t_0\mathbf{1}+t_1p_1+\cdots+t_rp_r+\sum_{\deg\alpha>2}t_\alpha\phi_\alpha.$$

We write ∂_α for the derivative $\frac{\partial}{\partial t_\alpha}$ and we write $\eta_{\epsilon\epsilon'}=\langle\phi_\epsilon,\phi_{\epsilon'}\rangle$. Note that if $\eta^{\epsilon\epsilon'}$ is the inverse matrix, then

$$\sum_{\epsilon\epsilon'}\eta^{\epsilon\epsilon'}\phi_\epsilon\otimes\phi_{\epsilon'}=PD(\Delta)$$

where $\Delta\subset X\times X$ is the diagonal.

We defined the *quantum product* $\circ$ by its structure constants:

$$\langle\phi_\alpha\circ\phi_\beta,\phi_\gamma\rangle:=F_{\alpha\beta\gamma}:=\partial_\alpha\partial_\beta\partial\gamma F.$$

The content of the WVDD equations is that $\circ$ is an associative product. The product depends upon the parameters t and q and is a deformation of the cup product in the sense that the quantum product and the cup product agree when $q=t=0$. When we set $t=0$, the product is called the *small quantum product*.

It is not necessarily the case that we can recover all the Gromov-Witten invariants (namely F) from the structure constants of the quantum product. The product structure alone does not "remember" all the information of the potential. The theme of the next two lectures will be to see what additional structure the quantum product has that "remembers" all the information of the GW invariants.

There are some hints that the extra structure should take the form of a differential equation. For example the structure constants $F_{\alpha\beta\gamma}$ came to us as derivatives: the quantum product should satisfy some sort of "integrability" condition. The other key source of the hint is from mirror symmetry. We will see in the first section an example where physics has predicted the GW-invariants of a certain Calabi-Yau 3-fold in terms of the solution to a certain differential equation (which

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Notes taken by J. Bryan.

[1]For notational convenience we will assume that X has no cohomology in odd degrees.

comes from a calculation involving variation of Hodge structure on the "mirror manifold".)

We begin in the first section with some general remarks about the finiteness of quantum cohomology and we give some examples, including the key example from mirror symmetry. In the second section, we build some machinery to formalize the extra structure present in the quantum product in terms of connections and differential equations.

1. Finiteness and examples

The (small) quantum cohomology ring $QH^*(X)$ is determined by a set of genus 0 GW-invariants on X that may be much smaller than the set of all genus 0 GW-invariants. For example if $c_1(X)$ is positive (X is *Fano*) then

$$\deg(q_i) = c_1(X)(p_i) > 0$$

and there are only a finite number of GW-invariants to compute for $QH^*(X)$, namely the part where

$$\deg(q^d) \leq 2\dim_{\mathbb{C}} X.$$

If in addition, $H^*(X)$ is generated by $H^2(X)$ then the Kontsevich-Manin reduction theorem tells us that we can recover the full genus 0 potential F from $QH^*(X)$ using the WDVV equation. Thus in this case *we can solve all of the genus 0 enumerative problems on X from the knowledge of only a finite number of them.* The original example of this phenomenon is $\mathbb{CP}^2$ where Kontsevich used the WDVV equation and the sole fact that there is one line through two points to recursively compute the number of rational curves in degree d passing through $3d-1$ points.

The opposite end of the spectrum from Fano is when $c_1(X)$ is negative. In this case there are only finitely many enumerative problems to begin with since the expected dimension of the moduli space of stable maps is

$$\mathrm{RRdim}(X_{0,0,d}) = c_1(X)(d) + \dim_{\mathbb{C}} X - 3$$

which is negative except for finitely many degrees.

However in general $c_1(X)$ can be positive on some classes and negative on others. One particularly poignant example is when $c_1(X) = 0$ (X is *Calabi-Yau* or just *CY*); this is the case important in "mirror symmetry". The simplest place to look for CY-manifolds of dimension n are as hypersurfaces in $\mathbb{P}^{n+1}$. A simple calculation shows that c_1 vanishes if and only if the degree of the hypersurface is $2+n$.

Example. A degree 3 curve $E \subset \mathbb{P}^2$ is an elliptic curve. Since π_2 is 0 for a smooth elliptic curve, there are no non-trivial holomorphic spheres.

Example. A degree 4 surface $X_{(4)} \subset \mathbb{P}^3$ is a $K3$ surface. By a small deformation of the Kähler structure on $X_{(4)}$ we obtain a $K3$ surface that has no holomorphic curves at all, and so all the GW-invariants are 0. Another way to see this is to observe that the Riemann-Roch dimension of all the genus 0 stable map moduli spaces is -1. This suggests that although the curves on $X_{(4)}$ vanish under generic perturbations of the almost complex structure, they may persist in one dimensional families[2].

[2]In fact, there is a one dimensional family of Kähler structures on $K3$ given by a hyperkähler structure (the "twistor" family) and it is unique up to deformation. One can compute GW-invariants in this family and finds that the answer can be expressed in terms of modular forms.

Example. A degree five 3-fold $X^3_{(5)} \subset \mathbb{P}^4$ is a key example coming from mirror symmetry. In this case the Riemann-Roch dimension of genus 0 maps is zero so that we expect genus zero curves on X to be rigid (requiring no constraints).

By the Lefschetz hyperplane theorem and Serre duality, all the even cohomology of X comes from $\mathbb{P}^4$ and we denote the generators by $1, p, p^2$, and p^3. The dimension of $H^3(X)$ is large (204) but does not contribute to the structure constants (by a simple dimension count). The only relevant structure constant is thus:

$$\begin{aligned} \langle p \circ p, p \rangle &= 5 + \sum_{d=1}^{\infty} q^d (p,p,p)_{0,3,d} \\ &= 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1-q^d} \end{aligned}$$

where n_d is the number of geometric rational curves in degree d (as oppose to maps, which is what the GW-invariants count). We make a short digression to explain the last equality.

The number of geometric curves differs from the GW-invariants in this case because of the possibility of multiple covers: a given geometric curve of degree d will have a non-trivial contribution to number of degree ld maps. This phenomenon is made possible because $c_1(X)$ evaluates as zero on this class; ordinarily multiple covers are ruled out because the Riemann Roch dimension of a multiple cover differs from the embedding.

There is a further complication: although the Riemann Roch dimension of $X_{0,0,d}$ is zero, the component of $X_{0,0,d}$ coming from multiple covers (of say a curve of degree d/l if $l|d$) has actual dimension $2l-2$ (since it can be identified with $\mathbb{P}^1_{0,0,l}$). The contribution thus has to be computed by evaluating the Euler class of the obstruction bundle on this moduli space. The answer of this computation was long predicted to be $1/l^3$ by physicists and was first computed rigorously by Kontsevich (with some combinatorial help from Manin). Assuming this computation, let's derive the above inequality. First

$$\sum_{d=1}^{\infty} q^d (p,p,p)_{0,3,d} = \sum_{d'=1}^{\infty} q^{d'} (d')^3 ()_{0,0,d'}$$

by the divisor equation (we changed indices from d to d' for later convenience). Now $()_{0,0,d'}$ is the unconstrained GW-invariant that counts "number of rational maps of degree d'" . Since for every $l|d'$ and every geometric curve of degree d'/l we get a contribution of $1/l^3$, we can relate $()_{0,0,d'}$ to the number of geometric curves as follows:

$$()_{0,0,d'} = \sum_{l|d'} \frac{1}{l^3} n_{d'/l}\,.$$

This will be the subject of my talk in the Joint Symplectic Geometry seminar on Oct 6 which you are all encouraged to attend! -J.B.

Recombining we get:

$$\begin{aligned}
\sum_{d=1}^{\infty} q^d (p,p,p)_{0,3,d} &= \sum_{d'=1}^{\infty} q^{d'} (d')^3 \sum_{l|d'} l^{-3} n_{d'/l} \\
&= \sum_{d'=1}^{\infty} q^{d'} \sum_{l|d'} (\frac{d'}{l})^3 n_{d'/l} \\
&= \sum_{d=1}^{\infty} d^3 n_d \sum_{l=1}^{\infty} q^{dl} \\
&= \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1-q^d}.
\end{aligned}$$

We used the substitution $d' = dl$ in the above.

We return to the example. Since the moduli spaces $X_{0,0,d}$ all have virtual dimension 0, the only non-zero contributions of the potential $F(q,t)$ come from two dimensional cohomology classes, *i.e.* $t = t_1 p$. We compute the potential function:

$$\begin{aligned}
F(t,q) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d=0}^{\infty} q^d (t,\ldots,t)_{0,n,d} \\
&= 5t_1^3 + \sum_{d=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} q^d d^n t_1^n ()_{0,0,d} \\
&= 5t_1^3 + \sum_{d'=1}^{\infty} e^{d' t_1} q^{d'} ()_{0,0,d'} \\
&= 5t_1^3 + \sum_{d'=1}^{\infty} (e^{t_1} q)^{d'} \sum_{l|d'} l^{-3} n_{d'/l} \\
&= 5t_1^3 + \sum_{d=1}^{\infty} n_d \sum_{l=1}^{\infty} l^{-3} (q e^{t_1})^{ld}.
\end{aligned}$$

We see that all the genus 0 GW-invariants are contained in the sequence $\{n_d\}$, but the potential F above satisfies the WDVV equations for any choice of the numbers $\{n_d\}$ (it satisfies WDVV rather trivially because there is only one t variable). Thus WDVV does not help us to answer any enumerative questions[3].

The numbers n_d were predicted by string theorists using the famous "mirror conjecture"[4]. While we do not describe the mirror conjecture or the physicist's calculation, we give here their description of the predicted values for n_d.

[3] A geometric reason for this is the following. The WDVV equations come from degenerating a rational curve to more than one component and considering the different distributions of the marked points. However, on a CY 3-fold, all the rational curves are rigid, so there is no degeneration!

[4] The numbers n_d have been recently rigorously computed in Givental's "Equivariant Gromov-Witten invariants" providing in particular the first examples where the mirror conjecture has been rigorously verified. -J.B.

Consider the following power series in t with coefficients in $\mathbb{Q}[p]/(p^4)$:

$$\begin{aligned} f &= e^{pt}\sum_{d=0}^{\infty} e^{dt}\frac{(5p+1)(5p+2)\cdots(5p+5d)}{(p+1)^5\cdots(p+d)^5} \\ &= f_0 + f_1 p + f_2 p^2 + f_3 p^4 \end{aligned}$$

so for example

$$f_0 = \sum_{d=0}^{\infty} \frac{e^{dt}(5d)!}{(d!)^5}.$$

The f_i's are a basis for the solutions to the 4th order O.D.E.

$$D^4 f = 5e^t(5D+1)(5D+2)(5D+3)(5D+4)f$$

where $D = \frac{d}{dt}$. Note that we could rewrite f and the equation with the variable $q = e^t$ and $D = q\frac{d}{dq}$. This differential equation has three singular points[5] at $0, 5^{-5}$, and ∞. Since f_0 is positive we can divide by f_0 and consider the functions

$$(1, \frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_4}) = (1, \tau, \phi_1(\tau), \phi_2(\tau))$$

as the solutions to a differential equation of the form

$$\left(\frac{\phi''(\tau)}{K(e^\tau)}\right)'' = 0 \tag{1}$$

where K is the function

$$K(q) = 5 + \sum_{d=1}^{\infty} m_d \frac{d^3 q^d}{1-q^d}.$$

The coefficients m_d are determined uniquely by the differential equations. The explicit solutions to Equation 1 are given by (the coefficients in the expansion in p of) the equation

$$e^{p\tau}\left(1 + \frac{p^2}{5}\sum_{d=1}^{\infty} m_d \sum_{l=1}^{\infty} \frac{e^{ld\tau}}{(p+ld)^2}\right) \mod p^4.$$

The mirror conjecture states that the coefficients $\{m_d\}$ that are determined by the above equations are the same as the coefficients $\{n_d\}$ that count the number of rational curves of degree d on $X^3_{(5)}$ and appear in the Gromov-Witten potential.

¿From our point of view, the origin of the above equations is completely mysterious. However, elements of the above equations are suggestive and motivate our exploration of the structure of the quantum product in terms of differential equations.

[5] I don't really know what this means. -J.B.

2. Flat connections, differential equations and the quantum product

In this section we lay the foundations for our understanding of the quantum product in terms of differential equations. We show that the quantum product defines a family of flat connections on the tangent bundle to the parameter space (the "(q,t)" space). We explicitly construct a basis of sections that are covariantly flat with respect to the family of connections. The construction turns out to involve the "gravitational descendants".

Recall that the WDVV equations state that the expression

$$\sum_{\epsilon\epsilon'} F_{\alpha\beta\epsilon}\eta^{\epsilon\epsilon'}F_{\epsilon'\gamma\delta}$$

is totally symmetric in the indices α, β, γ, and δ. It implies in particular that the quantum product defined by

$$\langle \phi_\alpha \circ \phi_\beta, \phi_\gamma \rangle$$

is associative. However, just saying that WDVV gives us associativity never uses the fact that the $F_{\alpha\beta\gamma}$'s are derivatives. In other words, the quantum product has some sort of integrability condition built in. To formalize this notion, recall that the parameter space for F (and $\circ$) is

$$H = H^*(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z}).$$

A priori the parameter space for the t variables is $H^*(X,\mathbb{C})$ and the parameter space for the q variables is $H^2(X,\mathbb{Z})$ but the divisor equation tells us that F only depends on $t_1,\dots,t_r$ and $q_1,\dots,q_r$ in the combinations $q_1e^{t_1},\dots,q_re^{t_r}$. We can thus absorb the q dependence into the t dependence which doesn't change with the addition of integral multiples of $2\pi i$.

We show how the quantum product defines a compatible system of connections on the tangent bundle of H. Since the tangent bundle of H is canonically trivialized (with fiber $H^*(X,\mathbb{C})$) we can think of sections of TH as just a $H^*(X,\mathbb{C})$ valued function on H. We denote a section S so that $S(t) \in H^*(X,\mathbb{C})$.

A connection on a bundle $E \to M$ is an operator

$$\nabla : C^\infty(E) \to C^\infty(E)\otimes\Omega^1(M)$$

satisfying the Leibnetz rule. In a local trivialization of E, sections can be thought of as (vector valued) functions and a connection can be written (locally) as

$$\nabla s = ds + As$$

where A a matrix valued one form. Even though A depends on the local trivialization, the *curvature tensor* is globally defined and can be expressed locally as

$$R = dA + A\wedge A.$$

A connection with $R=0$ is called a *flat* connection.

In our case, the bundle $TH \to H$ is globally trivialized so that a connection is defined by a global 1-form A on H with values in $\mathrm{End}(H^*(X,\mathbb{C}))$.

We use the quantum product to define a family of flat connections ∇_h on TH depending on a parameter $h \neq 0$. Define

$$A_h = \frac{1}{h}\sum_\alpha dt_\alpha A_\alpha := \frac{1}{h}\sum_\alpha dt_\alpha \phi_\alpha \circ .$$

In order to see that this connection is flat, we must see that $dA + A \wedge A$ vanishes. Since the two terms are of different order in $1/h$, it must be that dA and $A \wedge A$ vanish separately.

Exercise. Show that $A \wedge A = 0$ is equivalent to the associativity of $\circ$.

SOLUTION: The $dt_\alpha \wedge dt_{\alpha'}$ term of $A \wedge A$ is the commutator $\frac{1}{h}[A_\alpha, A_{\alpha'}]$ (recall that each A_α is a matrix mapping $H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$) and so this tells us that the A_α's commute. Now the quantum product is by definition *commutative*, so we show associativity as follows:

$$\begin{aligned}(\phi_\alpha \circ \phi_\beta) \circ \phi_\gamma &= \phi_\gamma \circ (\phi_\alpha \circ \phi_\beta) \\ &= A_\gamma(A_\alpha(\phi_\beta)) \\ &= A_\alpha(A_\gamma(\phi_\beta)) \\ &= \phi_\alpha \circ (\phi_\gamma \circ \phi_\beta) \\ &= \phi_\alpha \circ (\phi_\beta \circ \phi_\gamma).\end{aligned}$$

In addition to associativity, flatness encodes the fact that the structure constants $F_{\alpha\beta\phi}$ are derivatives:

Exercise. Show that $dA = 0$ is equivalent to $F_{\alpha\beta\phi} = \partial_\alpha F_{\beta\gamma}$.

SOLUTION: We write out the matrix indices of A so that

$$A = \frac{1}{h} \sum_\alpha A^\gamma_{\alpha\beta} dt_\alpha.$$

Each matrix element is a 1-form and $dA = 0$ says that they are all closed one forms. If our underlying space H were simply connected, DeRham theory would tell us that A is exact so that $A = dF$ or

$$\sum_\alpha A^\gamma_{\alpha\beta} dt_\alpha = \sum_\alpha \partial_\alpha (F^\gamma_\beta)\, dt_\alpha$$

for some functions F^γ_β. Because H is not simply connected, F may in general be a circle-valued function[6]

It turns out that the family of connections ∇_h is consistent in the sense that there are sections S that are covariantly flat for all h ($\nabla_h S = 0$). Said another way, it means we can solve the system of equations

$$h\partial_\alpha S = \phi_\alpha \circ S. \tag{2}$$

We give some examples where we can explicitly solve Equation 2.

Example. Suppose X is a single point. The only variable in this case is t_0 and Equation 2 is $h\partial_{t_0} S = S$. We can solve this by $Ce^{t_0/h}$ for some constant C.

Example. Suppose that X is $\mathbb{CP}^1$. Then our independent variables are t_0 and $q_1 e^{t_1}$ and our dependent variable is $S = s_0\mathbf{1} + s_1 p_1$. Equation 2 breaks into two equations

$$\begin{aligned}h\partial_0 S &= S \\ h(\partial_1 s_0 + \partial_1 s_1 p) &= s_0 p + s_1 q e^{t_1}\mathbf{1}\end{aligned}$$

[6] If we knew that the 1-form A pulled back from a 1-form on the universal cover of H, then we can conclude that F^γ_α are honest functions. Perhaps this is the case here. -J.B.

where we used $p \circ p = qe^{t_1}\mathbf{1}$. The first equation just says that the dependence of S on t_0 is by an overall multiplicative factor of $e^{t_0/h}$. The second equation is a first order system that by replace by a single second order equation using the substitution $s_0 = \partial_1 s_1$:

$$h^2 \frac{d^2 s_1}{dt_1^2} = qe^{t_1} s_1.$$

We can write the two solutions $\{s_1^{(0)}, s_1^{(1)}\}$to this equation as the coefficients in the p expansion of a kind of hyper-geometric series:

$$s_1^{(0)} + s_1^{(1)} p = e^{pt/h} \sum_{d=0}^{\infty} \frac{e^{dt} q^d}{(p+h)^2 (p+2h)^2 \cdots (p+dh)^2} \mod p^2.$$

Example. Finally, let us return to the example of the Calabi-Yau quintic 3-fold $X^3_{(5)}$.

Recall that the only interesting structure constant in this case is

$$\begin{aligned} \langle p \circ p, p \rangle &= \partial_1^3 F \\ &= 5 + \sum_{d=1}^{\infty} n_d \sum_{l=1}^{\infty} d^3 (qe^{t_1})^{ld} \\ &= 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 (qe^{t_1})^d}{1 - (qe^{t_1})^d} \\ &= K(qe^{t_1}) \end{aligned}$$

so that

$$p \circ p = p^2 (1 + \frac{1}{5} \sum_{d=1}^{\infty} n_d \frac{d^3 (qe^{t_1})^d}{1 - (qe^{t_1})^d}) = K(qe^{t_1}) \frac{p^2}{5}. \tag{3}$$

Let's look at the part of Equation 2 that involves the ∂_1 derivative:

$$h(\partial_1 s_0 + \partial_1 s_1 p + \partial_1 s_2 p^2 + \partial_1 s_3 p^3) = s_0 p + s_1 K(qe^{t_1}) \frac{p^2}{5} + s_2 p^3.$$

If we denote the ∂_1 derivative by a dot, the above equation is the system:

$$\begin{aligned} \dot{s}_0 &= 0, \\ \dot{s}_1 &= s_0, \\ \dot{s}_2 &= \frac{s_1}{5} K(qe^{t_1}), \\ \dot{s}_3 &= s_2, \end{aligned}$$

which by successive substitution we see that the solutions $\{s_0, \ldots, s_3\}$ are the independent solutions to

$$\left(\frac{\ddot{s}_3}{K(qe^{t_1})} \right)^{\cdot\cdot} = 0.$$

Note that this equation also appeared in the mirror symmetry formulation.

We now endeavor to write down general solutions to Equation 2. Eventually we hope to imitate the examples and convert the first order system into a single higher order equation.

Theorem 1. *For each fixed cohomology class ϕ_ν,*

$$S = \sum_{\mu\gamma} S_{\mu\nu}\eta^{\mu\gamma}\phi_\gamma$$

solves

$$h\partial_\alpha S = \phi_\alpha \circ S$$

where $S_{\mu\nu}$ is defined by

$$S_{\mu\nu} := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d\in\Lambda} q^d(\phi_\mu, t, \ldots, t, \frac{\phi_\nu}{h-c})_{0,n+2,d}.$$

For the $d = n = 0$ term in the above expression we take as its definition $\langle\phi_\mu, \phi_\nu\rangle = \eta_{\mu\nu}$.

Proof: We need two key facts:

1. $\phi_\alpha \circ (\partial_\beta S) = \phi_\beta \circ (\partial_\alpha S)$,
2. $h\partial_0 S = S$.

The theorem follows from these two facts by setting the index β to 0 in the first equation and applying the second:

$$\begin{aligned} h\partial_\alpha S &= h\mathbf{1} \circ (\partial_\alpha S) \\ &= h\phi_\alpha \circ (\partial_0 S) \\ &= \phi_\alpha \circ S. \end{aligned}$$

We show how the first fact follows from the composition law and the second follows from the string equation. Note that from the definition of $S_{\mu\nu}$, the effect of taking the derivative ∂_α is to replace one of the t's with ϕ_α. For example, the second fact follows by applying the string equation to each term after taking the t_0 derivative:

$$\begin{aligned} (\phi_\mu, 1, t, \ldots, t, \frac{\phi_\nu}{h-c}) &= (\phi_\mu, t, \ldots, t, D\frac{\phi_\nu}{h-c}) \\ &= \frac{1}{h}(\phi_\mu, t, \ldots, t, \frac{\phi_\nu}{h-c}). \end{aligned}$$

This fails if we start with $(\phi_\mu, 1, \frac{\phi_\nu}{h-c})$, but this is just $\frac{1}{h}\langle\phi_\mu, \phi_\nu\rangle$ which coincides with our definition.

For the proof of the first fact we will use the summation convention that repeated greek indices are summed over. We want to show that the expression

$$\phi_\beta \circ (\partial_\alpha S_{\mu\nu}\eta^{\mu\gamma}\phi_\gamma)$$

is symmetric in α and β. We take its inner product with an arbitrary element ϕ_δ and show that the result is symmetric in α and β:

$$\langle\phi_\beta \circ (\partial_\alpha S_{\mu\nu}\eta^{\mu\gamma}\phi_\gamma), \phi_\delta\rangle = (\partial_\alpha S_{\mu\nu})\eta^{\mu\gamma}F_{\beta\gamma\delta}.$$

Now the the WDVV equation exactly says that the right hand side of the equation would be totally symmetric in α, ν, β, and δ if $S_{\mu\nu}$ were replaced by $F_{\mu\nu}$. In fact, $S_{\mu\nu}$ is the same as $F_{\mu\nu}$ except that ϕ_ν has been replaced by $\frac{\phi_\nu}{h-c}$. Therefore the same proof of the WDVV equation applies except that the symmetry now only applies to α, β, and δ because of the special role of the ϕ_ν term in $S_{\mu\nu}$. In particular the right hand side is symmetric in α and β.