TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 25

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In the previous lecture we gave a heuristic outline of how doing "$\frac{1}{2}$−infinite dimensional Morse theory on the universal cover $\widetilde{\mathcal{L}\mathcal{X}}$ of the loop space $\mathcal{L}\mathcal{X}$ leads to a $\mathcal{D}$-module structure on $H^*(X;\mathbb{Q}[q^{\pm},h^{\pm}])$. As $h\to 0$ the heuristic tells us that we recover the quantum cohomology ring. We applied the heuristics to a naïve compactification of a finite dimensional approximation to the loop space $X_{(d)}$. We discovered the appearance of the same hypergeometric series $I(q,h)$ that appeared in the physicists formulation of mirror symmetry. One rigorous result that was obtained from the calculation with $X_{(d)}$ is the curious result:

Lemma 1. *The series $I(q,h)$ (defined in the previous lecture) has the following "polynomial property":*

$$\left\langle I(qe^{hz},h), I(q,-h)\right\rangle$$

is a power series in h (as oppose to h^{-1}).

Recall that we are working to prove our mirror theorem (Theorem **??**) which says that the series J (which encodes relations in the quantum cohomology ring) is related to the hypergeometric series I by a triangular change of coordinates. In some cases, the theorem says that I and J agree on the nose which indicates that in those cases the naïve compactification $X_{(d)}$ of the loop space leads to the correct answer. In general, however, I and J differ by a change in coordinates which indicates that the naïve compactification is not the correct one. In any case, we need to use the stable maps to compute quantum cohomology so we should seek a version of the loop space picture using stable maps.

Definition 1. Let $\mathcal{X}_{(n),d}$ denote the moduli space $X\times\mathbb{P}^1_{0,n,(d,1)}$ of n−marked, genus 0 maps to $X\times\mathbb{P}^1$ in the class $(d,1)$.

$\mathcal{X}_{(n),d}$ is a compactification of the space of degree d algebraic loops since we can consider the graph of a degree d "loop" $\mathbb{P}^1\to X$. Furthermore, S^1 acts on $\mathcal{X}_{(n),d}$ since it acts on the second factor of $X\times\mathbb{P}^1$.

Our heuristic tells us that we should consider S^1-equivariant Gromov-Witten theory on $X\times\mathbb{P}^1$. The ordinary S^1-equivariant cohomology of $X\times\mathbb{P}^1$ is

$$H^*_{S^1}(X\times\mathbb{P}^1) = H^*(X;\mathbb{Q}[\rho,h]/\rho(\rho-h))$$

since $H^*_{S^1}(\mathbb{P}^1)=\mathbb{Q}[\rho,h]/\rho(\rho-h)$ (here $H^*_{S^1}(\text{pt.})=\mathbb{Q}[h]$).

By applying localization we will prove the following theorem.

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Notes taken by J. Bryan.

Theorem 1. *Let* $T = \sum_{i=1}^r p_i \ln q_i + \rho \sum_{i=1}^r p_i z_i$ *and let* $(T, \dots, T)^{S^1}_{\mathcal{X}_{(n),d}}$ *be the corresponding* S^1*-equivariant Gromov-Witten invariant of* $X \times \mathbb{P}^1$*. Then*

$$\sum_{d \in \Lambda} \sum_{n=0}^{\infty} \frac{1}{n!} (T, \dots, T)^{S^1}_{\mathcal{X}_{(n),d}} = \left\langle J(qe^{hz}, h), J(q, -h) \right\rangle.$$

Although we obtained this result by following the heuristic loop space approach to quantum cohomology, the only role of this theorem in the actual proof of the mirror theorem is the following corollary (analogous to Lemma 1):

Corollary 1. $\left\langle J(qe^{hz}, h), J(q, -h) \right\rangle$ *is a power series in* h*.*

Theorem 1 (and the corollary) are valid for arbitrary X given the usual caveats concerning the existence of a virtual fundamental class and its properties. The theorem also holds in the context of the supermanifold ΠE. Recall that if $E \to X$ is a holomorphic bundle then the total space can be given the structure of a supermanifold ΠE. The appropriately defined version of Gromov-Witten invariants for ΠE can be substituted for the Gromov-Witten invariants of the complete intersection $s^{-1}(0)$ for a section $s : X \to E$. If $E \to X$ is convex then we get a bundle $\mathcal{E}_{n,d} \to \mathcal{X}_{(n),d}$ with fiber $H^0(\Sigma, f^*(E))$ over $f : \Sigma \to X \times \mathbb{P}^1$. We define a fundamental cycle by its evaluation on cocycles in $\mathcal{X}_{(n),d}$:

$$\int_{[\Pi \mathcal{E}_{n,d}]} \theta := \int_{[\mathcal{X}_{n,d}]} \theta \wedge \mathrm{Euler}_{S^1}(\mathcal{E}_{n,d})$$

with a similar definition for the intersection pairing $\langle \cdot, \cdot \rangle_{\Pi E}$. Recall that in this context

$$J_{\Pi E} = e^{\mathbf{p} \ln \mathbf{q}/h} \left(1 + \frac{1}{h} \sum_{d \in \Lambda, d \neq 0} q^d ev_* \left(\frac{\mathrm{Euler}(\widetilde{\mathcal{E}}_{1,d})}{h - c} \right) \right)$$

where $ev_* : X_{1,d} \to X$ is the evaluation map and $\mathcal{E}_{1,d} \to X_{1,d}$ is the bundle with fiber $H^0(\Sigma, f^*(E) \otimes \mathcal{O}(-x_1))$ over $f : \Sigma \to X$ ($x_1 \in \Sigma$ is the marked point).

Theorem 1 holds for ΠE and in particular we have

Corollary 2. $\langle J_{\Pi E}(qe^{hz}, h), J_{\Pi E}(q, -h) \rangle_{\Pi E}$ *is a power series in* h*.*

We sketch the proof of Theorem 1. The localization formula tells us that $(T, \dots, T)^{S^1}_{\mathcal{X}_{n,d}}$, which is computed as in integral over $\mathcal{X}_{n,d}$, can be computed as an integral over the S^1 fixed point set of $\mathcal{X}_{n,d}$.

What are the fixed points in $\mathcal{X}_{n,d}$? If $\{0\}$ and $\{\infty\}$ are the two fixed points in $\mathbb{P}^1$ then stable maps to $X \times \mathbb{P}^1$ that are fixed must have fixed image. Thus if $f \in \mathcal{X}_{n,d}$ is a fixed point, then its image must be a union of curves in $X \times \{0\}$ and $X \times \{\infty\}$ and a single line $\{\mathrm{pt.}\} \times \mathbb{P}^1$. The curves in $X \times \{0\}$ and $X \times \{\infty\}$ have degrees d' and d'' satisfying $d' + d'' = d$ and they have n' and n'' marked points satisfying $n' + n'' = n$. Thus the fixed point set in $\mathcal{X}_{n,d}$ can be described in terms of moduli spaces of stable maps on X:

$$(\mathcal{X}_{d,n})^{\mathrm{fixed}} = \bigcup_{d'+d''=d, n'+n''=n} X_{n'+1,d'} \times_{\Delta} X_{n''+1,d''}$$

where Δ is the constraint that the extra marked point on each side agree (this encodes the point $\{x_0\}$ where the curves meet $\{x_0\} \times \mathbb{P}^1$).

To use the localization formula, we must also compute the normal bundle of $(\mathcal{X}_{n,d})^{\text{fixed}} \subset \mathcal{X}_{n,d}$. The fiber of the normal bundle over a fixed curve

$$C' \times \{0\} \cup \{x_0\} \times \mathbb{P}^1 \cup C'' \times \{\infty\}$$

is the four dimensional space

$$T\mathbb{P}^1|_{\{0\}} \oplus T\mathbb{P}^1|_{\{\infty\}} \oplus (T\mathbb{P}^1|_{\{0\}} \otimes TC'|_{\{x_0\}}) \oplus (T\mathbb{P}^1|_{\{\infty\}} \otimes TC''|_{\{x_0\}})$$

corresponding to the deformations

$$\begin{aligned} C' \times \{0\} &\mapsto C' \times \{\epsilon\} \\ C'' \times \{\infty\} &\mapsto C'' \times \{\infty + \epsilon\} \end{aligned}$$

and the smoothing of the points $\{x_0\} \times \{0\}$ and $\{x_0\} \times \{\infty\}$.

We see that the first Chern classes of the four line bundle factors are h, $-h$, $h - \pi_1^*(c)$, and $-h - \pi_2^*(c)$ respectively where π_1 and π_2 are the projections of $X_{n'+1,d'} \times_\Delta X_{n''+1,d''}$ to the first and second factors.

To compute how the class T restricts to $(\mathcal{X}_{n,d})^{\text{fixed}}$ observe that ρ restricts to the class $(0,h) \in H^*_{S^1}(X_{n'+1,d'} \times_\Delta X_{n''+1,d''})$. Thus

$$T|_{(\mathcal{X}_{n,d})^{\text{fixed}}} = (\sum_i p_i \ln q_i, \sum_i p_i(\ln q_i + z_i h)) = (t, \tau).$$

Applying the localization theorem expresses $(T, \ldots, T)^{S^1}_{\mathcal{X}_{n,d}}$ as a sum of integrals over $X_{n'+1,d'} \times_\Delta X_{n''+1,d''}$ of $ev_1^*(T) \wedge \cdots \wedge ev_n^*(T) \wedge \text{Euler}^{-1}(\text{normal})$. These integrals can be expressed in terms of the ordinary Gromov-Witten invariants of X by integrating on each of the factors $X_{n',d'}$ and $X_{n'',d''}$ separately.

The result is

$$\sum_{d \in \Lambda} \sum_{n=0}^{\infty} \frac{1}{n!} (T, \ldots, T)^{S^1}_{\mathcal{X}_{n,d}} = \sum_{n',n'',d',d''} \frac{1}{h(n')!} \left(\frac{\phi_\alpha}{h - c}, t, \ldots, t \right) \eta^{\alpha\beta} \frac{1}{h(n'')!} \left(\tau, \ldots, \tau, \frac{\phi_\beta}{-h - c} \right).$$

The diagonal condition $\Delta \subset X_{n'+1,d'} \times X_{n''+1,d''}$ is expressed in the above formula by summing over the basis ϕ_α and its dual basis $\eta^{\alpha\beta}\phi_\beta$. The restriction of the inverse Euler class to each factor is responsible for the $h^{-1}(\pm h - c)^{-1}$ terms and the restriction of T to the factors is responsible for the t's and the τ's. The choices for splitting the n points into sets of size n' and n'' leads to the $n'!$ and $n''!$ factors.

By repeated applications of the divisor equation we can rewrite the RHS of the equation as

$$\langle J(e^t, h), J(e^\tau, -h) \rangle$$

and derive the equation of the theorem.

Only one slight amendment to the above discussion must be made and that is when d' or d'' is 0. In these cases the normal bundle is no longer four dimensional and the summation on the right of the localization formula should not have the corresponding $1/h$ factor for those terms. This is compensated for in the definition of J: the $d = 0$ term of J has no $1/h$ factor. □

To carry through the heuristic picture of equivariant Floer theory on the loop space, we have used two different finite dimensional models for loop space, namely $X_{(d)}$ and $\mathcal{X}_d$ leading to the series I and J respectively. Let us examine in the case of $X = \mathbb{P}^n$ the possible source of differences between I and J.

There is a map

$$\mathcal{X}_d \to \mathbb{P}^{nd+n-1} = \mathbb{P}^n_{(d)}$$

given as follows. A curve $C \in \mathcal{X}_d = (\mathbb{P}^n \times \mathbb{P}^1)_{(d,1)}$ has one component C_0 that is a graph of a degree d_0 map $\mathbb{P}^1 \to \mathbb{P}^n$ and so is given by relatively prime, degree d_0 polynomials $[x_0(w) : \cdots : x_n(w)]$ (here w is the homogeneous coordinate on $\mathbb{P}^1$). The remaining components of C are degree d_i maps $\mathbb{P}^1 \to \mathbb{P}^n \times \{w_i\}$ where $w_1, \ldots, w_k$ are points in $\mathbb{P}^1$ and $\sum_{i=0}^{k} d_i = d$. We get an element in $\mathbb{P}^n_{(d)}$ by

$$[x_0(w) : \cdots : x_n(w)](w - w_1)^{d_1} \cdots (w - w_k)^{d_k}.$$

The map defined[1] above identifies the fundamental classes of $\mathcal{X}_d$ and $\mathbb{P}^n_{(d)}$ and so we expect the series I and J for $\mathbb{P}^n$ to coincide. However, in the case of a convex bundle $V \to \mathbb{P}^n$ (if we are interested in a complete intersection in $\mathbb{P}^n$) the Gromov-Witten invariants use a fundamental class defined by the Euler classes of vector bundles $V_d \to \mathbb{P}^n_{(d)}$ and $\mathcal{E}_d \to \mathcal{X}_d$. It is not necessarily the case that the map $\mathcal{X}_d \to \mathbb{P}^n_{(d)}$ pulls back the bundle V_d to the bundle $\mathcal{E}_d$. The difference accounts for the difference between I and J.

[1] Strictly speaking, this map is not well-defined when w_i is 0 or ∞ for some i. The set where this occurs is codimension at least 2 and so perhaps the rational map is enough to identify the fundamental classes. This point should be clarified.