TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 24

A. GIVENTAL

1. Floer cohomology.

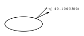
Quantum cohomology was introduced by Floer in his paper on symplectic topology. Symplectic topology is a Morse theory for abstract functionals on the loop space. Let (X, ω, J) be a Kahler manifold. Let LX be the space of contractible loops $S^1 \to X$.

1. LX inherits the symplectic structure on X. The symplectic form on the loop space is defined as follows:

$$\Omega(\xi, \eta) = \int_{S^1} \omega(\xi(t), \eta(t)) dt$$

Tangent space = infinitesimal deformations of a loop.

Figure 1. Two vector fields at a point of a loop space.

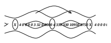


2. We have a natural S^1 action on LX induced by translations on S^1. . The action is obviously symplectic and the Hamiltonian H is defined to be action functional $\mathcal{A}(\gamma) = \int\int \omega : \tilde{LX} \to \mathbb{R}$. The integral is taken over the disk which has the loop γ as a boundary and as an answer gives the area of the disk inside the loop. It obviously depends on the choice of the disk, so the Hamiltonian is a multivariable function in LX defined up to the areas of spheres (symplectic periods of ω). Therefore it is natural to consider H as a single-valued function on the covering $\tilde{LX}$ rather then LX.

Exercise. Check it is a Hamiltonian.

Let us assume that $\pi_1(X) = 0$. Then $\pi_1(LX) = \mathbb{Z}$ and $\tilde{LX}$ is a $\mathbb{Z}$-covering of LX. Figure 2 shows the action of the covering transformation q. In the finite-dimensional

Figure 2. Action of the covering transformation.



case we had remarkable localization theorem for equivariant cohomology. Unfortunately, it does not apply directly to infinite-dimensional spaces. Our plan is to construct the new cohomology, called Floer cohomology, which plays the role of equivariant cohomology in the infinite-dimensional setting.

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Notes taken by Alexander Kogan.

The Floer cohomology FH^* is constructed using Morse theory on LX with Morse function being the action functional H. The key observation is that the gradient flows are the analytic continuations of loops to holomorphic cylinders.

Floer cohomology is hard to define rigorously, but is not hard to compute. It is equal to the equivariant cohomology of the critical set of H, which is just a bunch of copies of X located at different levels of $\tilde{LX}$:

$$FH^*_{S^1}(\tilde{LX}) = H^*(X, \mathbb{Q}[[q, \hbar]]),$$

where q is a covering transformation as above and $\hbar$ is a generator in $H^*(\mathbb{C}P^\infty) = \mathbb{Q}[\hbar]$.

It turns out that Floer cohomology has a structure of a $\mathcal{D}$-module. Let's describe it in more details. Let us use for that the equivariant DeRham cohomology

$$\Omega^*(\tilde{LX})^{S^1}[\hbar],\ D = d + \hbar i_{S^1},$$

where i_{S^1} is the contraction with the velocity vector field on S^1. Next we introduce two operations on differential forms. Let $q = q^*$ be the pullback via the covering transformation q. Let $p = (\omega + \hbar H)\wedge$.

Proposition 1.
- $[p, q] = pq - qp = \hbar q$.
- $[\hbar \frac{d}{dt}, e^t] = \hbar e^t$.

Conclusion : $H^*(X, \mathbb{Q}[[q, \hbar]])$ is a module over the algebra $\mathcal{D}$ of differential operators generated by $\hbar\frac{d}{dt}, e^t$.

Example. $X = \mathbb{C}P^{n-1}$. Let [1] be the form dual to the fundamental class. The $\mathcal{D}$-module is generated by [1] and the relations are

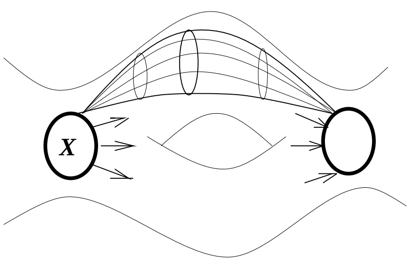
$$p^n[1] = q[1].$$

So the Floer cohomology is just

$$\mathbb{C}[p, q]/(p^n - q)$$

where $pq - qp = \hbar q$.

FIGURE 3. The picture of the gradient flows.



Remark. We see that in the limit $\hbar \to 0$ we obtain just the quantum cohomology of $\mathbb{C}P^{n-1}$.

2. Coputation of the integrals.

We wanted to compute $\int_{[1]} e^{Pz}$ last time. Let us approximate [1] by $[1]_d$ - holomorphic curves in X of degree at most d.

$$\int_{[1]_d} e^{pz} \to \int_{[1]} e^{pz}$$

In the previous lecture we saw that $[1]_d = \mathbb{C}P^{5d+4}$.

Suppose we have the usual setting as in the Figure: and let X be the corre-

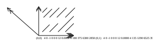
Figure 4. The setting for the reduction to toric manifold.

sponding toric manifold,

$$X = \mathbb{C}^n / T^r.$$

Let $\phi : \mathbb{C}P^1 \to X$ be a generic holomorphic map of degree d. We can lift ϕ to a map $\tilde{\phi} : \mathbb{C}P^1 \to \mathbb{C}^n$. The image is given by n polynomials of degree less or equal to $D_i(d) = \sum d_i m_{ij}$. The space of coefficients $\mathbb{C}^{n+\sum D_j(d)}$ has torus action and we have the following picture: Thus we get another toric manifold $X_{(d)}$ - compactification

Figure 5. Torus action on the space of coefficients.



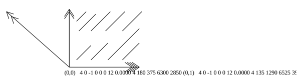
of the space of maps of degree d from $\mathbb{C}P^1$ to X.

Example. Let $X = X_1$ - toric manifold from the Example 1 of the previous lecture. We obtained from reduction of $\mathbb{C}\mathbf{P}^5$ by 2-dimensional torus using the matrix

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Recall that in $\mathbb{R}^2$ we had 5 vectors - images of basis vectors in $\mathbb{R}^5$. To get the picture corresponding to $X_{(d)}$ we have to repeat i-th vector $D_i(d)$ time, omitting negative D_i's. For example, the picture below shows the case $d = (2,1)$ (compare with Figure 1 of the Lecture 23).

Figure 6. Image of basis vectors of $\mathbb{R}^8_+$ in $\mathbb{R}^2$.



The Kahler cone $\bar{K}$ is the intersection of the images in $\mathbb{R}^r$ of r-faces in $\mathbb{R}^n_+$. The semigroup Λ of degrees of compact holomorphic curves in X coincides with the convex hull $K^\vee$ of the r-faces in $\mathbb{R}^{r*}$ dual to the images of r-faces.

Denote

$$K^\vee = \bigcup_{\text{vertices}} \text{dual of the images of } r\text{-faces} \subset H_2(X, \mathbb{Z}).$$

We want to consider

$$\sum_{d\in K^\vee} q^d \int_{[X_{(d)}]} e^{p_1z_1+\cdots+p_rz_r} Euler_{S^1}(V_{(d)})Euler_{S^1}(\text{ obstruction bundle })$$

The bundles entering the above integral will be defined below.

Suppose we want to study the Gromov-Witten theory on some submanifold $Y \subset X$, which is a complete intersection in X of codimention l. It means that Y is the zero locus of holomorphic sections of l line bundles. We obtain these line bundles over X in a following way. Let $\mathbb{C}\times\mathbb{C}^n$ be the trivial line bundle over $\mathbb{C}^n$. We extend the action the torus T^n to this line bundle by choosing some character of the torus. Then we quotient by the action of T^r and get a line bundle over X.

Thus we will consider pair (X, V), where V is a convex rank l bundle over X. We want to construct a vector bundle $V_{(d)}$ over $X_{(d)}$. Let $\phi : \mathbb{C}\mathbf{P}^1 \to X$ be a point in $X_{(d)}$. Then the fiber of $V_{(d)}$ at ϕ is defined to be $H^0(\mathbb{C}\mathbf{P}^1, \phi^*(V))$.

The obstruction bundle on $X_{(d)}$ is defined as follows: its fiber at the point ϕ is equal to $H^1(\mathbb{C}P^1, \phi^*(\text{ Normal bundle }))$.

Theorem 1. *(application of the fixed point localization theorem)*

$$\sum_{d\in K^\vee} q^d \int_{[X_{(d)}]} e^{p_1z_1+\cdots+p_rz_r} \mathrm{Euler}_{S^1}(V_{(d)})\mathrm{Euler}_{S^1}(\textit{ obstruction bundle }) =$$

$$< \vec{I}(qe^{z\hbar}, 1/\hbar), \vec{I}(q, 1/\hbar) >,$$

where $<,>$ *is the intersection pairing in the cohomology of complete intersections, defined as* $<\phi,\psi>=\int_{[X]}\phi\psi\mathrm{Euler}V$.

Corollary. $\vec{I} = e^{p\ln q/\hbar}(\text{ series in } q, 1/\hbar)$, but $< \vec{I}(qe^{z\hbar}, 1/\hbar), \vec{I}(q, 1/\hbar) >$ is a series in variables $q, z, \hbar$ and not $1/\hbar$.