Prelimanary version

TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 29

A. GIVENTAL

Our goal for today is to sketch some applications of localization technique to genus one G-W invariants.

1. Genus One Gromov-Witten Invariants

Define the genus one potential:

$$G(t) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \in \Lambda} q^d (t, t, \cdots, t)_{1,n,d},$$

where $t = \sum t_\alpha \phi_\alpha$ and $(n,d) = (0,0)$ term is defined to be zero. In genus one case, what really matters is not the function $G(t)$ but rather its differential (the reason will be clear later) :

$$d_t G(t) = \sum_\alpha dt_\alpha \sum_{n=0}^{\infty} (\phi_\alpha, t, \cdots, t)_{1,n+1,d}. \tag{1}$$

Now we want to express $dG(t)$ in terms of the fundamental solution

$$S_{\alpha i} =< \phi_\alpha, \phi_i > + \sum_{(n,d)\neq(0,0)} \frac{q^d}{n!} (\phi_\alpha, t, \cdots, t, \frac{\phi_i}{\hbar - c})_{0,n+2,d}.$$

In *singularity theory* (cf. lectures 14-16) we have Morse function $f_t(x)$ for generic t. The extra data we have are:

$u_1, \cdots, u_N$: critical points,

$\Delta_1, \cdots, \Delta_N$: Hessians at critical points,

$R_1, \cdots, R_N$: $\hbar$ order term of diagonal components of the oscillating integrals, ($R_i = R_{ii}$ in previous notation,

$\omega_t^{m,0}$: primitive form (of Saito).

The partial derivative of the oscillating integral is:

$$\hbar \frac{\partial}{\partial u_i} \int_{\Gamma_i} e^{f_{t(u)}(x)/\hbar} \omega_{t(u)}^{m,0}$$
$$\approx \hbar^{m/2} \,(\text{const})\, e^{u_i/\hbar}(1 + \hbar R_i + o(\hbar)) \qquad \text{(stationary phase asymptotics).}$$

Proposal.

$$dG := \frac{1}{48} \sum_i \frac{d\Delta_i(u)}{\Delta_i(u)} + \frac{1}{2} \sum_i R_i(u) du_i.$$

Question. Does dG satisfy all axioms of $g = 1$ Gromov-Witten theory?

So far, no one has checked this. Now we will give the motivation of this proposal in the context of *Gromov-Witten* theory.

Date: Dec. 4, 1997.
Notes taken by Y.-P. Lee.

Motivation. Given a concave vector bundle $E \to X$ with a torus T action such that X has isolated zero and one dimensional orbits. Let us define

$U := diag(u_1, \cdots, u_N)$, u_i are canonical coordinates,

$E := diag(e_1, \cdots, e_N)$, e_i are the equivariant Euler classes $Euler_T(T_i E)$,

$\delta_1, \cdots, \delta_N$: p_i-type potentials, $e_i \delta_i^2 = \Delta_i$, see lecture 28 for definition.

By theorem 2 in lectures 15-16 (in fact, slightly modified version in the case of concave vector bundle), we have

$$(S_{\alpha i}) = (\Psi_\alpha^j)(\delta_{ji} + \hbar R_{ji} + o(\hbar)) e^{U/\hbar} E^{-1}. \tag{2}$$

Compare this to the recursion relation we found in lecture 28,

$$\begin{aligned} S_{\alpha i} e^{-u_i/\hbar} e_i &=: s_{\alpha i}(\hbar) \\ &= \delta_{\alpha i} + \sum_{j \neq i} \sum_{m=1}^{\infty} \frac{C = (\equiv 0 \ \ mod\ q)}{x_i^j + m\hbar}, \end{aligned} \tag{3}$$

where the numerator C has no poles at $\hbar = 0$ and therefore can be expanded as power series in $\hbar$. Define $R_{ii} := R_{ii}$inequation$(2) \equiv 0\,(mod\ q)^1$.

Theorem 1. *Under usual assumptions, we have*

$$d_t G(t) = \frac{1}{24} \sum \frac{d\delta_i}{\delta_i} - \frac{1}{24} \sum_i c_{-1}^{(i)} du_i + \frac{1}{2} \sum R_{ii} du_i, \tag{4}$$

where

$$\begin{aligned} c_{-1}^{(i)} :&= \frac{c_{top-1}(T_i E)}{c_{top}(T_i E)} \\ &= \sum_{\chi:\ chracters\ of\ T\ :\ T_i E} \frac{1}{\chi} \end{aligned}$$

and c_* are the equivariant chern classes.

Remark. A similar formula holds for convex super-manifolds, but $c_{-1}^{(i)}$ should be defined differently.

The relation of this theorem (in G-W theory) and the proposal (in singularity theory) is[2]:

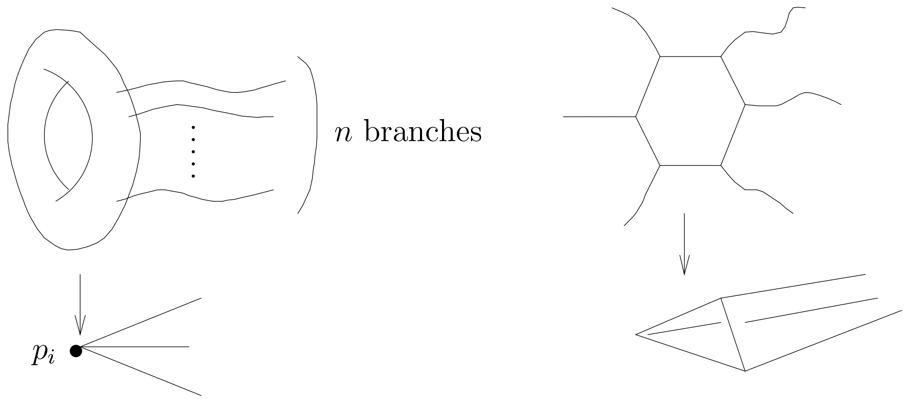
$$d_t G = \frac{1}{48} \sum \frac{d\Delta_i}{\Delta_i} + \frac{1}{2} \sum \underbrace{(R_{ii} - \frac{1}{12} c_{-1}^{(i)})}_{R_i} du_i.$$

Conjecture. The proposal is true for generic semisimple Frobenius structure of compact symplectic manifolds.

Proof. (of the theorem) (Skech.) The proof will again be based on fixed point localization technique. The fixed points of $X_{1,n,d}$ can be divided into two types. The first type has an elliptic curve with n branches out, and the second type contains one cycle of rational components:

[1] R_{ii} are defined only up to constants. But the recursion relation can fix these constants.

[2] Here we might have to readjust the constant of R_i. But in all worked examples $R_i = R_{ii} - \frac{1}{12} c_{-1}^{(i)}$.



Case 1 case 2

Fig. 1

Our claim now is that the contribution of case 1 is the first two terms of (4) and that of case 2 is the last term.

Case 1. The contribution of fixed points of case 2 type to dG is:

$$e_i^{-1}\sum_{n=0}^{\infty}\int_{\overline{\mathcal{M}_{1,n}}}(1-c_{-1}^{(i)}\omega)T(c^{(1)})\wedge\cdots\wedge T(c^{(n)})\int_{\overline{\mathcal{M}_{0,*}}}\cdots\int\cdots. \tag{5}$$

Here $T(c)$ is some function of chern class as usual. ω is the first chern class of the Hodge bundle $\mathcal{H}$ and the term $(1-c_{-1}^{i}\omega)$ comes from the obtruction part. The obstruction is of the form $H^1(C_e,T_iE)=T_iE\otimes\mathcal{H}^*$ (C_e is the irreducible component of genus one). The contribution of this to dG is of the form:

$$\begin{aligned}\frac{Euler_T(T_iE\otimes\mathcal{H})}{Euler_T(T_iE)}=&\frac{chern_T(T_iE)(-\omega)}{Euler_T(T_iE)}\\ &(chern_T(T_iE)(-\omega)\text{ is the chern polynomial with variable equal to }-\omega)\\ =&\frac{c_{top}+c_{top-1}(-\omega)}{c_{top}}\\ =&1-c_{-1}^{(i)}\omega.\end{aligned}$$

Apart from the obstruction part, it looks very much like the potential $\nu(T)=\frac{1}{24}\ln\delta_i(T)$ (. The extra part from obstruction is of the form $c_{-1}^{(i)}\mu(T)$ (see lecture 28 for definitions). Therefore we conclude that case 1 part is equal to $\frac{1}{24}\ln\delta_i-\frac{1}{24}c_{-1}^{(i)}u_i$ as claimed.

Case 2. If we cut the cycle in case 2 (Fig. 1) at a vertex which maps to fixed point p_i, we will end up with (for G)

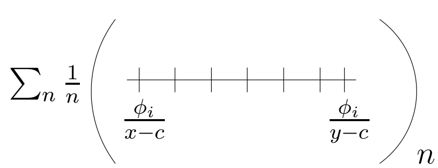


Fig. 2

Here the factor $1/n$ comes from the choice of the cut vertices. But this series is of the form like log function $(\sum \frac{1}{n})$, it is not very convenient for computation. Therefore we consider the partial derivative $\frac{\partial G}{\partial t_\alpha}$:

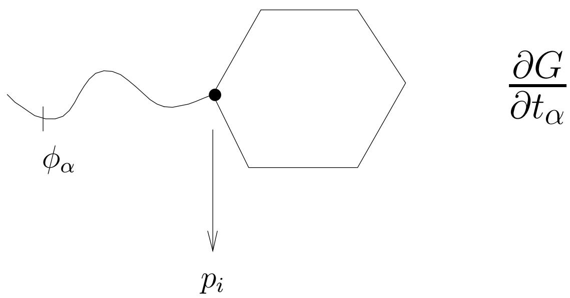


Fig. 3

Then this part is related to the potential

$$V_{ij}(x,y) = \sum_{n=0} \frac{q^d}{n!}(\frac{\phi_i}{x-c}.t, \cdots, t, \frac{\phi_j}{y-c})_{0,n+2,d}$$

which we defined in lecture 10. If we take the limit:

$$\frac{1}{2}\frac{\partial u}{\partial t_\alpha} \lim_{x,y\to 0}[V_{ii}(x,y)e^{-[(u_i/x)+(u_i/y)]}e_i - \frac{1}{x+y}] \tag{6}$$

which is just $\frac{\partial G}{\partial t_\alpha}$ (Fig. 3). From an exercise in lecture 10, we have:

$$V_{ij}(x,y) = \frac{1}{x+y}\sum_\alpha S_{\alpha i}(x)e_\alpha^{-1}S_{\alpha j}(y). \tag{7}$$

Replace $\hbar$ in (2) by x, y respectively and substitute into (7). Use the obvious identity

$$\sum_\alpha \Psi^i_\alpha e_\alpha^{-1}\Psi^j_\alpha = e_i\delta_{ij}$$

we then have

$$V_{ii} = \frac{e^{u_i(\frac{1}{x}+\frac{1}{y})}}{e_i(x+y)}[1 + R_{ii}(x+y) + \cdots]$$

$$\Rightarrow \lim_{x,y\to 0}[V_{ii}e^{u_i/x+u_i/y}e_i - \frac{1}{x+y}] = R_{ii}.$$

Combined with (6) this then implies that dG in case 2 is equal to $\sum_i \frac{1}{2}du_iR_{ii}$ as claimed.

$\square$