TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 28

A. GIVENTAL

1. Quantum Potentials on $\overline{\mathcal{M}}_{g,n}$

Let us first introduce some potentials which we will use to

$$u(T) = \sum_{n=1}^{\infty} \frac{1}{n!} < 1, T(c), \cdots, T(c), 1 >_{n+2}, \tag{1}$$

$$s(T, \hbar) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} < 1, T(c), \cdots, T(c), \frac{1}{\hbar - c} >_{n+2}, \tag{2}$$

$$v(T, x, y) = \frac{1}{x+y} + \sum_{n=1}^{\infty} \frac{1}{n!} < \frac{1}{x-c}, T(c), \cdots, T(c), \frac{1}{y-c} >_{n+2}, \tag{3}$$

$$\delta(T) = \sum_{n=0}^{\infty} \frac{1}{n!} < 1, 1, 1, T(c), \cdots, T(c), \frac{1}{y-c} >_{n+3}, \tag{4}$$

$$\mu(T) = \sum_{n=1}^{\infty} \frac{1}{n!} \omega[T(c), \cdots, T(c)]_n, \tag{5}$$

$$\nu(T) = \sum_{n=1}^{\infty} \frac{1}{n!} [T(c), \cdots, T(c)]_n, \tag{6}$$

where ω is the first chern class of the Hodge bundle and

$$T(c) = c + t_1 c + t_2 c^2 + \cdots,$$

$$< \cdots >_n := \int_{\overline{\mathcal{M}}_{0,n}} \cdots,$$

$$\omega[\cdots]_n := \int_{\overline{\mathcal{M}}_{1,n}} \omega \cdots,$$

$$[\cdots]_n := \int_{\overline{\mathcal{M}_{1,n}}} \cdots.$$

In order to make sense of our potentials, some requirements on the convergence property of $T(c)$ are necessary. In our case, we will require that $T(c) = \sum_d q^d R_d(c)$ where $R_d(c)$ is a rational function of c with no pole at 0 and $R_0 = constant$. Thus $t_0 \in \mathbb{Q}[[q]], t_1, t_2, \cdots \in (q)\mathbb{Q}[[q]]$. The main properties of these potentials are summarized in the following propostion:

Date: Dec. 2, 1997.

Notes taken by Y.-P. Lee.

Proposition 1.

$$s = e^{u/\hbar}, \qquad v = \frac{e^{u(\frac{1}{x}+\frac{1}{y})}}{x+y},$$
$$\delta = \frac{1}{1-t_1}, \qquad \mu = \frac{u}{24},$$
$$\nu = \frac{\ln\delta}{24}.$$

Proof. Define
$$\mathcal{L} := \partial_0 - t_1\partial_0 - t_2\partial_1 - t_3\partial_2 - \cdots .$$
It's easy to see that $\mathcal{L}T(c) = 1 - \frac{T(c)-T(0)}{c}$. By string equation $\mathcal{L}F$ is "essentially" zero (F is any of the above potential), except for some beginning terms in genus zero or one. The precise statements are:

Lemma 1. *$\mathcal{L}\delta = 0$, $\mathcal{L}\nu = 0$, $\mathcal{L}u = 1$, $\mathcal{L}\mu = \frac{1}{24}$, $\mathcal{L}s = \frac{s}{\hbar}$, and $\mathcal{L}v = (\frac{1}{x}+\frac{1}{y})v$.*

Thus we can consider u as the time variable of the "flow of the vector field" $\mathcal{L}$. By the above lemma, we only need the initial values of the potentials. Our potentials then look like:
$$\delta(u) = \delta(0), \nu(u) = \nu(0), \mu(u) = \mu(0) + \frac{u}{24},$$
$$s = e^{u/\hbar}s(0), v = e^{u(\frac{1}{x}+\frac{1}{y})}v(0).$$

Now our cliam is:

Claim. *If $T(c) = \sum q^d R_d(c) = t_0 + t_1 c + t_2 c^2 + \cdots$, ($t_n \in \mathbb{Q}[[q]]$) satisfies the reqirements stated in the beginning of the lecture, then the flow of $\mathcal{L}$ is well defined. Each trajectory (in the set of all such T's) crosses the plane $t_0 = 0$ exactly once.*

Proof. (*of the claim*) Let τ be the time. $t_n = t_n(\tau)$ is a fuction of τ. The flow of $\mathcal{L}$ is well defined bacause we can explicitly construct convergent power series for t_n:
$$t_0(\tau) = \tau + \sum_{n+0}^{\infty} t_n(0)\frac{(-\tau)^n}{n!},$$
$$t_1(\tau) = 1 - \frac{dt_0(\tau)}{d\tau},$$
$$t_i(\tau) = -\frac{dt_{i-1}(\tau)}{d\tau}, \quad i \geq 2.$$
The forms of $t_i(\tau)$ follow from the explicit form of $\mathcal{L}$, and the convergence of $t_0(\tau)$ (over $\mathbb{C}[[q]]$) follows from the conditions on $T(c)$ ($t_n(0)$ is a convergent power series in q) (exercise).

Remark. In fact, there are singular points of the flow, e. g. $\tau = t_0 + c$. But our conditions on $T(c)$ ecxclude these cases.

The second claim can be seen as follows. We want to find the solutions of

(7) $$0 = t_0(\tau) = \tau + \sum_{n+0}^{\infty} t_n(0)\frac{(-\tau)^n}{n!}.$$

Since $R_0(c)$ is constant, (7) is equal to $\tau + t_0(0) = 0 \pmod{q}$ (recall that $t_i|_{q=0} = 0$ is our condition). Therefore

$$\tau = -t_0(0) + \sum_{k+1}^{\infty} ?_k q^k.$$

Without loss of generality we may assume $t_0(0) = 0$. Then (7) gives a recursion relation of $?_k$. Hence, second claim is proved. □

As already noticed, we only need to find the initial values of these potentials (now we take u as time variable again). First for u. By dimension reason, $t_0 = 0$ implies $u = 0$. And the converse is also true by the claim. For δ:

$$\delta(0) = \sum_{n=0}^{\infty} \frac{1}{n!} < 1, 1, 1, t_1 c, \cdots, t_1 c >_{n+3} \tag{8}$$

where the terms involving $t_n c^n (n \geq 2)$ all vanish due to dimensional reason. By dilation equation, (8) is equal to

$$\begin{aligned} \delta(0) &= \sum (n t_1) < 1, 1, 1, t_1 c, \cdots >_{n+2} \\ &= \sum_{n=0}^{\infty} \frac{n! t_1^n}{n!} < 1, 1, 1 >_3 \\ &= \frac{1}{1 - t_1}. \end{aligned}$$

For ν:

$$\begin{aligned} \nu(0) &= \sum \frac{1}{n!} [t_1 c, \cdots, t_1 c]_n \\ &\overset{dilation}{=} \sum \frac{t_1^n (n-1)!}{n!} [c]_1 \\ &= \frac{1}{24} \sum_{n=1}^{\infty} \frac{t_1^n}{n} \\ &= \frac{1}{24} \ln \frac{1}{1 - t_1} \\ &= \frac{1}{24} \ln \delta. \end{aligned}$$

$\mu(0) = 0$ for dimension reason. For $s(0)$ and $v(0)$ all but initial term vanish, therefore $s(0) = 1$, $v(0) = \frac{1}{x+y}$. This concludes our proof. □

2. Local Composition Laws and Fixed-point Localization

Recall the Gromov-Witten potential in genus zero is defined to be:

$$F = \sum_{n,d} \frac{q^d}{n!} (t, \cdots, t)_{0,n,d},$$

and the quantum multiplication is defined by $F_{\alpha,\beta,\gamma}$. We will use degenaration method to get another presentation of quantum multiplication. Let us choose a generic cross ratio of five points on a component of a rational stable curve : (*Convention: in all the graphic presentations, we will always suppress the "exponential sums"*)

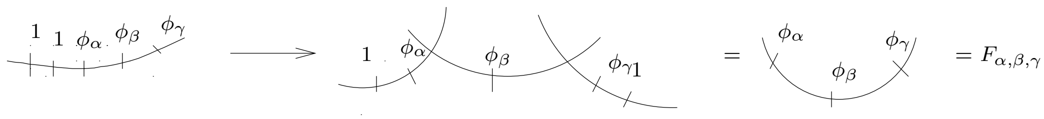


Fig. 1

We assume that the manifold X has a torus action with isolated fixed points p_α's and isolated one dimensional orbits. Let ϕ_α be the δ-functions at the fixed points normalized as $< \phi_\alpha, \phi_\beta >= \frac{\delta_{\alpha\beta}}{e_\alpha}$, where e_α is the (equivariant) Euler class of the tangent bundle at p_α (product of characters e. g. $e_\alpha = \prod_{\beta\neq\alpha}(\lambda_\alpha - \lambda_\beta)$ on $\mathbb{C}\mathbf{P}^n$). A general cohomology element is written as $t = \sum t_\alpha \phi_\alpha$

Observe that a stable map with $n+5$ marked points in a given generic configuration (i. e. with a given generic value of contraction map) must have an irreducible component C_s (special component) in the underlying curve C, which contains contains this given configuration (i. e. the marked points either sit on C_s or on a branch connecting to C_s such that these marked points and intersection points form the given configuration). This allows us to divide all fixed point components of stable maps into types p_i aacording to which fixed point C_s is mapped to. Thus $F_{\alpha,\beta,\gamma} = \sum_i F^{(i)}_{\alpha,\beta,\gamma}$ *(we will reserve the index i for the type p_i).* Here we remark that degree of $F^{(i)}_{\alpha\beta\gamma} = 3 - dim(X) - 3 - dim(X) = 0$. According to Fig.1 our $F_{\alpha\beta\gamma}$ can degenerate:

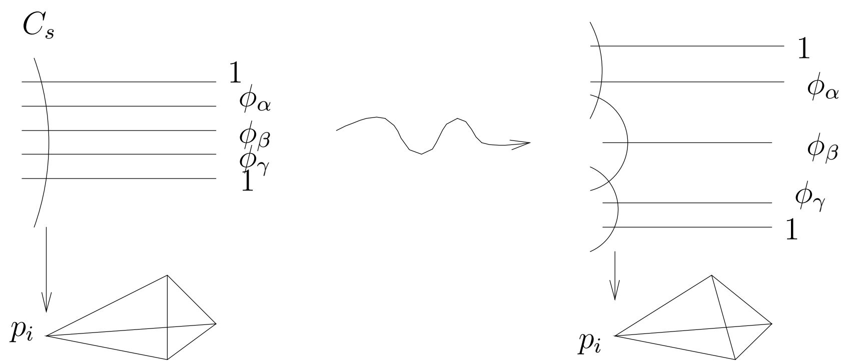
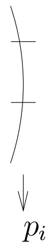
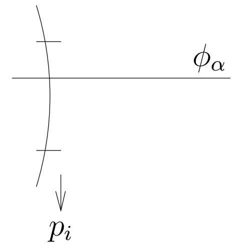
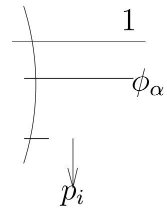
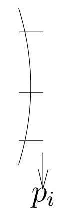


Fig. 2

The degeneration of generic configuration is, in fact, a diagonalization of the quantum multiplication $\phi_\beta \circ$.

Remark. There are other degenerate configurations which are not the limits of generic configurations. Our arguments show that they don't contribute to $F_{\alpha\beta\gamma}$.

We now define some quantities:

graph				
name	u_i	$\partial_\beta u_i$	Ψ^i_α	δ_i
mod q	t_i	$\delta_{\beta i}$	$\delta_{\alpha i}$	1
homog. deg.	1	0	0	0
contrib. to	$e_i\partial_i\partial_i F$	$e_i\partial_i\partial_i\partial_\beta F$	$e_i\partial_i\partial_\alpha\partial_0 F$	$e_i\partial_i\partial_i\partial_i F$

Theorem 1.

$$(1) F_{\alpha\beta\gamma} = \sum_i \Psi^i_\alpha \frac{\partial_\beta u_i}{e_i} \Psi^i_\gamma$$

$$(2) \sum_i \Psi^i_\alpha e_i^{-1} \Psi^i_\gamma = \delta_{\alpha\gamma} e_\alpha^{-1}$$

$$\sum_\alpha \Psi^i_\alpha e_\alpha \Psi^j_\alpha = \delta_{ij} e_i$$

$$(3) \sum_\alpha \Psi^i_\alpha = \delta_i^{-1}$$

$$(4) \Psi^i_\beta = (\sum_\alpha \Psi^i_\alpha)\partial_\beta u_i = \delta_i^{-1}\partial_\beta u_i$$

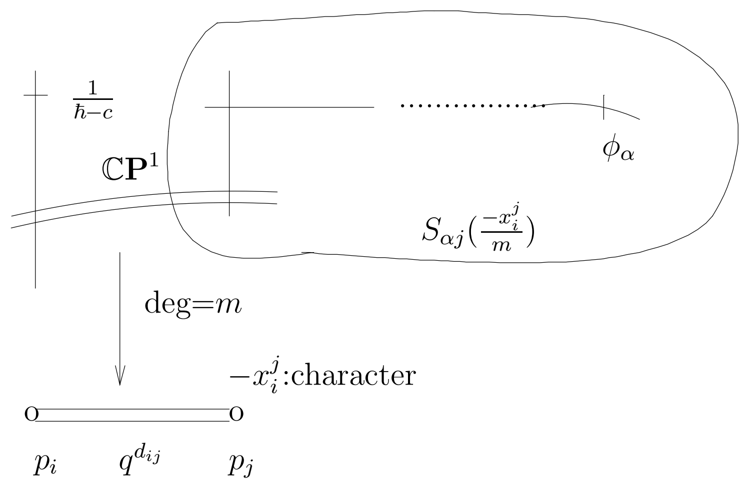
Corollary 1. *We have a materialization of Dubrovin's canonical coordinate theory, i. e. u_i are the canonical coordinates and $\Delta_i = e_i\delta_i^2$ is the Hessian in the singularity theoretic picture (cf. lecture 14, 15).*

3. Fundamental Solution Matrices in Canonical Coordinates

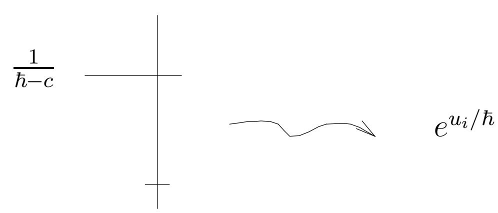
Recall our fundamental solution matrix is

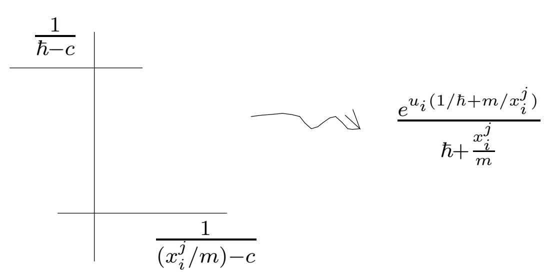
$$S_{\alpha i} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d\in\Lambda} q^d (\phi_\alpha, t, \cdots, t, \frac{\phi_i}{\hbar - c})_{0,n+2,d}.$$

Graphically,



Decompose:





Repeat the same derivation of the recursion relation in flat coordinates, we will arrive at the following:

Theorem 2.

$$s_{\alpha i}(\hbar) := e^{-u_i/\hbar} S_{\alpha i}(\hbar) e_i$$

$$s_{\alpha i}(\hbar) = \delta_{\alpha i} + \sum_{j \neq i} \sum_{m=1}^{\infty} \frac{q^{d_{ij} m} e^{\frac{(u_i - u_j) m}{x_i^j}}}{x_i^j + m\hbar} s_{\alpha j}(-\frac{x_i^j}{m}).$$

The recursion relation determines $(s_{\alpha i})$ unambiguously as a function of canonical coordinates.

Problem. Solve this recursion relation.