TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 28

A. GIVENTAL

1. Quantum Potentials on $\overline{M}_{g,n}$

Let us first introduce some potentials which we will use to

\begin{align*}
(1) \quad u(T) &= \sum_{n=1}^{\infty} \frac{1}{n!} \langle 1, T(c), \cdots, T(c), 1 \rangle >_{n+2}, \\
(2) \quad s(T, \hbar) &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \langle 1, T(c), \cdots, T(c), \frac{1}{\hbar - c} \rangle >_{n+2}, \\
(3) \quad v(T, x, y) &= \frac{1}{x + y} + \sum_{n=1}^{\infty} \frac{1}{n!} \langle \frac{1}{x - c}, T(c), \cdots, T(c), \frac{1}{y - c} \rangle >_{n+2}, \\
(4) \quad \delta(T) &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle 1, 1, T(c), \cdots, T(c), \frac{1}{y - c} \rangle >_{n+3}, \\
(5) \quad \mu(T) &= \sum_{n=1}^{\infty} \frac{1}{n!} \langle [T(c), \cdots, T(c)]_n \rangle, \\
(6) \quad \nu(T) &= \sum_{n=1}^{\infty} \frac{1}{n!} \langle [T(c), \cdots, T(c)]_n \rangle,
\end{align*}

where $\omega$ is the first Chern class of the Hodge bundle and

\begin{align*}
T(c) &= c + t_1 c + t_2 c^2 + \cdots, \\
\langle \cdots \rangle_n &= \int_{\overline{M}_{0,n}} \cdots, \\
\omega[\cdots]_n &= \int_{\overline{M}_{1,n}} \omega \cdots, \\
[\cdots]_n &= \int_{\overline{M}_{1,n}} \cdots.
\end{align*}

In order to make sense of our potentials, some requirements on the convergence property of $T(c)$ are necessary. In our case, we will require that $T(c) = \sum_d q^d R_d(c)$, where $R_d(c)$ is a rational function of $c$ with no pole at 0 and $R_0 = constant$. Thus $t_0 \in \mathbb{Q}[q], t_1, t_2, \cdots \in (q) \mathbb{Q}[q]$. The main properties of these potentials are summarized in the following proposition:

Notes taken by Y.-P. Lee.
Proposition 1.

\[ s = e^{u/h}, \quad v = e^{u(\frac{x}{2} + \frac{y}{2}) \over x+y}, \]
\[ \delta = {1 \over 1-t_1}, \quad \mu = {u \over 24}, \]
\[ \nu = \ln \delta \over 24. \]

Proof. Define \( L := \partial_t - t_1 \partial_t - t_2 \partial_1 - t_3 \partial_2 - \cdots. \)

It's easy to see that \( LT(c) = 1 - \frac{T(c) - T(0)}{c}. \) By string equation \( LF \) is “essentially” zero \((F\) is any of the above potential), except for some beginning terms in genus zero or one. The precise statements are:

**Lemma 1.** \( L \delta = 0, \) \( L \nu = 0, \) \( L u = 1, \) \( L \mu = {u \over 24}, \) and \( L s = s \over h, \) \( L v = (1 \over 2x + 1 \over 2y) v. \)

Thus we can consider \( u \) as the time variable of the “flow of the vector field” \( L. \) By the above lemma, we only need the initial values of the potentials. Our potentials then look like:

\[ \delta(u) = \delta(0), \nu(u) = \nu(0), \mu(u) = \mu(0) + {u \over 24}, \]
\[ s = e^{u/h} s(0), \quad v = e^{u(\frac{x}{2} + \frac{y}{2}) \over x+y} v(0). \]

Now our claim is:

**Claim.** If \( T(c) = \sum q^d R_d(c) = t_0 + t_1 c + t_2 c^2 + \cdots, \) \((t_n \in \mathbb{Q}[[q]])\) satisfies the requirements stated in the beginning of the lecture, then the flow of \( L \) is well defined.

Each trajectory (in the set of all such \( T \)'s) crosses the plane \( t_0 = 0 \) exactly once.

Proof. (of the claim) Let \( \tau \) be the time. \( t_n = t_n(\tau) \) is a function of \( \tau. \) The flow of \( L \) is well defined because we can explicitly construct convergent power series for \( t_n: \)

\[ t_0(\tau) = \tau + \sum_{n=0}^{\infty} t_n(0) {(-\tau)^n \over n!}, \]
\[ t_1(\tau) = 1 - \frac{dt_0(\tau)}{d\tau}, \]
\[ t_i(\tau) = -\frac{dt_{i-1}(\tau)}{d\tau}, \quad i \geq 2. \]

The forms of \( t_i(\tau) \) follow from the explicit form of \( L, \) and the convergence of \( t_0(\tau) \) (over \( \mathbb{C}[[q]] \)) follows from the conditions on \( T(c) \) \((t_n(0) \) is a convergent power series in \( q) \) (exercise).

**Remark.** In fact, there are singular points of the flow, e.g. \( \tau = t_0 + c. \) But our conditions on \( T(c) \) exclude these cases.

The second claim can be seen as follows. We want to find the solutions of

\[ 0 = t_0(\tau) = \tau + \sum_{n=0}^{\infty} t_n(0) {(-\tau)^n \over n!}. \]
Since \( R_0(c) \) is constant, (7) is equal to \( \tau + t_0(0) = 0 \pmod{q} \) (recall that \( t_i|_{q=0} = 0 \) is our condition). Therefore

\[
\tau = -t_0(0) + \sum_{k+1}^\infty ?_k q^k.
\]

Without loss of generality we may assume \( t_0(0) = 0 \). Then (7) gives a recursion relation of \(?_k\). Hence, second claim is proved. \( \square \)

As already noticed, we only need to find the initial values of these potentials (now we take \( u \) as time variable again). First for \( u \). By dimension reason, \( t_0 = 0 \) implies \( u = 0 \). And the converse is also true by the claim. For \( \delta \):

\[
\delta(0) = \sum_{n=0}^{\infty} \frac{1}{n!} < 1, 1, 1, t_1 c, \cdots, t_1 c >_{n+3}
\]

where the terms involving \( t_n c^n (n \geq 2) \) all vanish due to dimensional reason. By dilation equation, (8) is equal to

\[
\delta(0) = \sum (nt_1) < 1, 1, 1, t_1 c, \cdots >_{n+2}
\]

\[
= \sum_{n=0}^{\infty} \frac{n t_1^n}{n!} < 1, 1, 1 >_3
\]

\[
= \frac{1}{1 - t_1}.
\]

For \( \nu \):

\[
\nu(0) = \sum \frac{1}{n!} [t_1 c, \cdots, t_1 c]_n
\]

\[
de_{dilation} = \sum \frac{t_1^n (n-1)!}{n!} [c]_1
\]

\[
= \frac{1}{24} \sum_{n=1}^{\infty} \frac{t_1^n}{n
\]

\[
= \frac{1}{24} \ln \frac{1}{1 - t_1}
\]

\[
= \frac{1}{24} \ln \delta.
\]

\( \mu(0) = 0 \) for dimension reason. For \( s(0) \) and \( v(0) \) all but initial term vanish, therefore \( s(0) = 1, v(0) = \frac{1}{x+y} \). This concludes our proof. \( \square \)

2. LOCAL COMPOSITION LAWS AND FIXED-POINT LOCALIZATION

Recall the Gromov-Witten potential in genus zero is defined to be:

\[
F = \sum_{n,d} q^n \frac{d}{n!}(t, \cdots, t)_{0,n,d},
\]

and the quantum multiplication is defined by \( F_{a, \beta, \gamma} \). We will use degeneration method to get another presentation of quantum multiplication. Let us choose a generic cross ratio of five points on a component of a rational stable curve: (Convention: in all the graphic presentations, we will always suppress the “exponential sums”)

\[
\begin{align*}
F &= \sum_{n,d} q^n \frac{d}{n!}(t, \cdots, t)_{0,n,d},
\end{align*}
\]
We assume that the manifold $X$ has a torus action with isolated fixed points $p_\alpha$'s and isolated one dimensional orbits. Let $\phi_\alpha$ be the $\delta$-functions at the fixed points normalized as $\langle \phi_\alpha, \phi_\beta \rangle = \frac{\delta_{\alpha\beta}}{e_\alpha}$, where $e_\alpha$ is the (equivariant) Euler class of the tangent bundle at $p_\alpha$ (product of characters e. g. $e_\alpha = \prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)$ on $\mathbb{CP}^n$). A general cohomology element is written as $t = \sum t_\alpha \phi_\alpha$.

Observe that a stable map with $n+5$ marked points in a given generic configuration (i. e. with a given generic value of contraction map) must have an irreducible component $C_s$ (special component) in the underlying curve $C$, which contains contains this given configuration (i. e. the marked points either sit on $C_s$ or on a branch connecting to $C_s$ such that these marked points and intersection points form the given configuration). This allows us to divide all fixed point components of stable maps into types $p_i$ according to which fixed point $C_s$ is mapped to. Thus $F_{\alpha,\beta,\gamma} = \sum_i F^{(i)}_{\alpha,\beta,\gamma}$ (we will reserve the index $i$ for the type $p_i$). Here we remark that degree of $F^{(i)}_{\alpha,\beta,\gamma} = 3 - \text{dim}(X) - 3 - \text{dim}(X) = 0$. According to Fig.1 our $F_{\alpha,\beta,\gamma}$ can degenerate:

The degeneration of generic configuration is, in fact, a diagonalization of the quantum multiplication $\phi_\beta \circ \phi_\alpha$.

**Remark.** There are other degenerate configurations which are not the limits of generic configurations. Our arguments show that they don’t contribute to $F_{\alpha,\beta,\gamma}$.

We now define some quantities:
Theorem 1.

\begin{align*}
(1) F_{\alpha\beta\gamma} &= \sum_i \Psi^i_{\alpha} \frac{\partial \beta u_i}{\epsilon_i} \Psi^i_{\gamma} \\
(2) \sum_i \Psi^i_{\alpha} e_i^{-1} \Psi^i_{\gamma} &= \delta_{\alpha\gamma} e_{\alpha}^{-1} \\
\sum_{\alpha} \Psi^i_{\alpha} e_{\alpha} \Psi^j_{\alpha} &= \delta_{ij} e_i \\
(3) \sum_{\alpha} \Psi^i_{\alpha} &= e_i^{-1} \\
(4) \Psi^i_{\beta} &= \left( \sum_{\alpha} \Psi^i_{\alpha} \right) \partial \beta u_i = \delta_i^{-1} \partial \beta u_i
\end{align*}

Corollary 1. We have a materialization of Dubrovin’s canonical coordinate theory, i.e., \( u_i \) are the canonical coordinates and \( \Delta_i = e_i \delta_i^2 \) is the Hessian in the singularity theoretic picture (cf. lecture 14, 15).

3. Fundamental Solution Matrices in Canonical Coordinates

Recall our fundamental solution matrix is

\[ S_{\alpha i} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \in \Lambda} q^d(\phi_{\alpha}, t, \cdots, t, \frac{\phi_i}{\hbar - \epsilon})_{0, n+2, d}. \]

Graphically,
Repeat the same derivation of the recursion relation in flat coordinates, we will arrive at the following:
Theorem 2.

\[ s_{\alpha_i}(h) = e^{-u_i/k} S_{\alpha_i}(h) e_i \]

\[ s_{\alpha_i}(h) = \delta_{\alpha_i} + \sum_{j \neq i} \sum_{m=1}^{\infty} d_{ijm} e^{(u_i - u_j)m} \frac{x_i^m}{x_i + m\hbar} s_{\alpha_j}(-x_i^m). \]

The recursion relation determines \((s_{\alpha_i})\) unambiguously as a function of canonical coordinates.

Problem. Solve this recursion relation.