TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 27

A. GIVENTAL

1. The Proof of Mirror Theorem (conclusion)

Our first goal today is to prove the equivalence of $\vec{J}$ and $\vec{I}$.

Recall that we have already had (for $\vec{J}$) :

1). Recurrsion relations:

$$J_\alpha(\hbar, q) = \sum_{d\in\Lambda} Pol_{d,\alpha}(\frac{1}{\hbar})q^d + \sum_{\beta\neq\alpha}\sum_{m=1}^{\infty} \frac{q^m Coeff_\alpha^\beta(m)}{\lambda_\alpha - \lambda_\beta + m\hbar} J_\beta(\frac{\lambda_\beta - \lambda_\alpha}{m}, q). \tag{1}$$

2). Polynomiality:

$$\sum_\alpha J_\alpha(\hbar, qe^{\hbar z})e^{\lambda_\alpha z} J_\alpha(-\hbar, q)\frac{Euler_G(V_\alpha)}{Euler_G(T_\alpha)} = \sum_{d\in\Lambda} q^d PS_d(\hbar, z), \tag{2}$$

where the symbol PS is some power series in $\hbar$ and z.

3). Asymptotic condition:

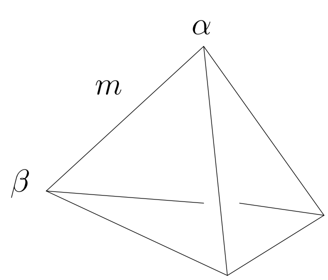
$$J_\alpha = 1 + o(\frac{1}{\hbar}). \tag{3}$$

Also it is known that I_α satisfies (1) and (2), but maybe not (3).

What we want to prove today is the following theorem:

Theorem 1. *$e^{(t_0+\lambda_\alpha \ln q)/\hbar} I_\alpha$ is transformed to $e^{(t_0+\lambda_\alpha \ln q)/\hbar} J_\alpha$ by changes of variables determined by the asymptotic property of I_α: $I_\alpha = A + \frac{B}{\hbar} + o(\frac{1}{\hbar})$.*

Proof. Our proof is divided into 2 steps. The first step is to prove that the changes of variables transform polynomial solutions, i. e. the changes of variables respect (2) (part 1) and (1) (part 2). The second step is to prove that a polynomial solution satisfying 3) is unique. A good example to keep in mind is the toric manifold $\mathbb{C}\mathbf{P}^3$:



Date: Nov. 25, 1997.
Notes taken by Y.-P. Lee.

We will illustrate the proof only for the change of variables :

$$\ln q \mapsto \ln q + f(q), \quad f(0) = 0,$$

which is the most complicated case. The others are left to the reader.

Step I (part 1) : Recall $q = e^\tau, t = \tau + \hbar z$. The change of variable:

$$q \mapsto q e^{f(q)} \tag{4}$$

$$z \mapsto z + \frac{f(qe^{\hbar z})}{\hbar}. \tag{5}$$

It is clear that series of q are mapped to series of q. But notice that the numerator in(5) is divisible by $\hbar$, therefore a power series in z is transformed to a power series in $z, q, \hbar$. We are done.

Step I (part 2):

$$\begin{aligned} J_\alpha &= e^{(-p\ln q)/\hbar} \vec{J}|_{p=\lambda_\alpha} \\ &\mapsto e^{\lambda_\alpha f(q)/\hbar} J_\alpha(\hbar, qe^{f(q)}) \\ &= \sum \widetilde{Pol} + \sum\sum \frac{q^m Coeff}{\lambda_\alpha - \lambda_\beta + m\hbar} J_\beta(\frac{\lambda_\beta - \lambda_\alpha}{m}, qe^{f(q)}) e^{\frac{\lambda_\alpha f(q)}{\hbar} + mf(q)} \end{aligned}$$

But the last exponent $\frac{\lambda_\alpha f(q)}{\hbar} + mf(q)$ at $\hbar = \frac{\lambda_\beta - \lambda_\alpha}{m}$ is equal to $\lambda_\beta f(q) \frac{m}{\lambda_\beta - \lambda_\alpha}$. The difference of these two is simply a polynomial and therefore can be absorbed into $\widetilde{Pol}$.

Remark. Step I is actually a consequence of divisor equation. Because the changes of variables are all of the form : $q \mapsto qe^t$.

Step II: This part of proof is based on *perturbation theory*. Suppose we have

$$\begin{cases} \widetilde{Pol}_{\alpha,m} = Pol_{\alpha,m} & \text{for } m < d \\ \widetilde{Pol}_{\alpha,d} = Pol_{\alpha,d} + R_\alpha(\frac{1}{\hbar}) \end{cases}$$

where R is the discrepency. By recurssion relation, $\widetilde{J_\alpha}$ and J_α coincide up to degree $d-1$ (in q) inclusively. The q^d term in (2)

$$\sum_\alpha e^{\lambda_\alpha z} [R_\alpha(\frac{1}{\hbar}) e^{d\hbar z} + R_\alpha(-\frac{1}{\hbar})] E_\alpha \tag{6}$$

is a power series in $\hbar, z$. Now expand (6) in z and $\hbar$: $\sum Const^k_{m,\alpha} z^k \hbar^m e^{\lambda_\alpha z} E_\alpha$. Since it is a power seires in $\hbar$, $Const^k_{m,\alpha} = 0$ for $m < 0$ by the linear independence of $1, z, z^2, z^3, \ldots$.

Conclusion:

$$[R_\alpha(\frac{1}{\hbar}) e^{d\hbar z} + R_\alpha(-\frac{1}{\hbar})] \tag{7}$$

is a power series (in $\hbar$) by itself.

Now our claim is: this conclusion will determine R_α uniquely up to the coefficients of 1 and $\frac{1}{\hbar}$. To see this write R as $A + B\hbar^{-1}$, A, B are polynomial functions of $\hbar^{-2}$. (7) is the same as

$$A(e^{d\hbar z} + 1) + B(\frac{e^{d\hbar z} - 1}{\hbar}) = 2A + zdB + o(\hbar^{-2k}).$$

This implies that if degree of A, B (in $\hbar^{-2}$) are less or equal to k, then degree of (7) is less than $k-1$ ((7) is a power series in $\hbar$). Thus A and B are constant. Therefore if $\widetilde{J} - J = O(\hbar^{-2})$, then they are equal. This concludes our proof. $\square$

2. Concave Vector Bundles

2.1. **Formulation.**

Definition. A concave bundle V (with total space E) over a (toric) manifold X is a vector bundle isomorphic to a direct sum of negative line bundles.

For example,

$$\begin{array}{c} V = \oplus_{j=1}^{k} \mathcal{O}(-l_j) \\ \Big\downarrow \\ \mathbb{C}\mathbf{P}^{n-1}. \end{array}$$

In this setting, we can also define the vector function $\vec{J}, \vec{I}$:

$$\vec{J}_E = e^{(t_0 + p\ln q)/\hbar}[1 + \frac{1}{\hbar}\sum_{d\neq 0} q^d ev_* \frac{Euler_T \widetilde{V'}_{1,d}}{\hbar - c}]$$

where ev is the evaluation map $X_{0,1.d} \to X$, $V'_{1,d}|_{(\Sigma,x;f)} = H^1(\Sigma, f^*V(-x)) = H^0(\Sigma, f^*V^* \otimes K_\Sigma(x))^*$, x is the marked point, c is the first chern class of the universal cotangent line.

$$\vec{I}_E = e^{(t_0 + p\ln q)/\hbar} \sum_{d=0}^{\infty} q^d \frac{\prod_{i=1}^{k} \prod_{m=0}^{l_i d - 1} (-l_i p + \lambda' - m\hbar)}{\prod_{m=1}^{d} (p + m\hbar)^n}$$

with $p^n = 0$.

Theorem 2. *$\vec{J}_E$ coincides with $\vec{I}_E$ up to change of variables.*

The proof of this theorem is the same as that of the convex vector bundle. We left it to the reader.

Corollary 1. *If $dim(V) = k > 1$, then $\vec{J}_E = \vec{I}_E$.*

Proof. Due of the appearance of $m = 0$ in the product of $\vec{I}_E$, it has the asymptotic property $1 + o(\frac{1}{\hbar})$. $\square$

Conjecture. (A. Givental) $\vec{J}_{\Pi E}$ coincides with $\vec{J}_{E^*}$ up to change of variables.

2.2. **Counting curves in quintic threefolds.**

First we consider the vector bundle $V \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{C}\mathbf{P}^1$. The primitive kähler class of $\mathbb{C}\mathbf{P}^1$ will be denoted by π, $\pi^2 = 0$. In this case

$$\begin{aligned} I_E &= e^{(\pi \ln q)/\hbar} \sum_{r=0}^{\infty} Q^r \frac{\prod_{m=0}^{r-1} (\pi - \lambda' + m\hbar)^2}{\prod_{m=1}^{r} (\pi + m\hbar)^2} \\ &= J_E \text{ (by the above corollary)} \end{aligned}$$

Notice that we change the notations a little bit and reserve the usual ones for the following case.

Consider another $\mathbb{C}\mathbf{P}^1$ embedded in a quintic Calabi-Yau threefold X with degree d. Denote p the hyperplane class of $\mathbb{C}\mathbf{P}^4$, and denote $q = e^t$ with t the coordinate of p. We have $p = d\pi$, $Q = q^d$ and $[\mathbb{C}\mathbf{P}^1]$ is Poincare dual to the cohomology class

dp^2. Generically the normal bundle to this embedded $\mathbb{C}\mathbf{P}^1$ is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Let E be the total space of the normal bundle. By our formulation

$$\begin{aligned}\int_{[E]} ? &= \int_{\mathbb{C}\mathbf{P}^1} \frac{1}{(\pi-\lambda')^2} ? \\ &= \int_X \frac{dp^2}{\pi-\lambda')^2} ? \\ &(p^4=0)\end{aligned}$$

So the $\vec{J}=\vec{I}$ for quintic X is equal to

$$\begin{aligned}\vec{J}_X &= e^{(p\ln q)/\hbar}[5+\sum_{d=1}^{\infty} dp^2 \sum_{r=1}^{\infty} \frac{q^{rd}}{(\frac{p}{d}+r\hbar)^2}] \\ &= e^{(p\ln q)/\hbar}[5+\sum_{d=1}^{\infty} d^3p^2 \sum_{r=1}^{\infty} \frac{q^{rd}}{(p+dr\hbar)^2}] \\ mod(p^4&=0)\end{aligned}$$

Now the explanation of the "strange procedure" proposed by Candelas *et. al.* is almost redundant. Denote

$$' = \hbar q \frac{d}{dq},$$

then

$$J'' = e^{(p\ln q)/\hbar} p^2 \underbrace{[5+\sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}]}_{K}$$

$$\left(\frac{J''}{K}\right)'' = p^4 = 0$$

which is exactly the equation proposed by [COGP].