TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 23

A. GIVENTAL

1. Illustration of the Mirror Theorem for toric manifolds.

Let X be a toric manifold which corresponds to the matrix M. Recall from the previous lecture that we had two generating functions $\vec{I}$, and $\vec{J}$ on X with values in the equivariant cohomology of X:

$$\vec{I} = e^{(t_0+pt)/\hbar} \sum_{d \in \Lambda} e^{dt} \frac{\prod_{m=\infty}^{0} (u_j(p) + m\hbar)}{\prod_{m=-\infty}^{D_j(d)} (u_j(p) + m\hbar)}$$

$$\vec{J} = e^{(t_0+pt)/\hbar} [1 + \frac{1}{\hbar} \sum_{d \neq 0} e^{dt} ev_* (\frac{1}{\hbar - c})]$$

The Mirror Theorem (lecture 21) states that $\vec{I} = \vec{J}$ after some change of variables.

There are two possible cases: $c_1(T_X) > 0$ (interior of the Kahler cone) and $c_1(T_X) \geq 0$ (boundary of the Kahler cone). The expansion of $\vec{I}$ in powers of $1/\hbar$ is different in these cases:

$$\vec{I} = e^{(t_0+pt)/\hbar} [1 + a_1 \hbar^{-1} (\text{in the first case } a_1 = 0) + o(\frac{1}{\hbar})].$$

Let us look closely at each particular case:

Case 1. $c_1(T_X)$ is in the interior of the Kahler cone.

In this case $\vec{I} = \vec{J}$. We don't need any change of variables. Therefore we get the following relations in quantum cohomology:

$$\prod_{j=1}^{n} u_j(p)^{m_{ij}} = q_i, i = 1, \ldots, r,$$

where $u_j(p) = \sum_{i=1}^{r} p_i m_{ij}$.

Case 2. $c_1(T_X)$ is on the boundary of the Kahler cone.

Here we possibly need some change of variables (we have to kill the term by $\hbar^{-1}$ in the expansion of $\vec{I}$).

Let us illustrate the second case in the examples of toric manifolds X_1 and X_2 constructed in the lecture 17.

Example. 1. Let X_1 be a 3-dimensional toric manifold corresponding to the matrix

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

X_1 is the projectivization of the sum of trivial and two Hopf line bundles over $\mathbb{C}P^1$.

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Notes taken by Alexander Kogan.

The image of the basis vectors under the projection $\mathbb{R}^5_+ \to \mathbb{R}^2$ used in the construction of X_1 is shown in Figure 1. It is clear that $c_1(T_{X_1}) = (0,3)$ is on the

FIGURE 1. Image of basis vectors of $\mathbb{R}^5_+$ in $\mathbb{R}^2$.

boundary of the Kahler cone (case 2).

Using Kirwan's theorem we compute the cohomology:

$$H^*(X_1) = \mathbb{Z}[p_1, p_2]/(p_1^2, p_2(p_2 - p_1)^2).$$

We compute the following formula:

$$\vec{I}_{(1)} = e^{(t_0 + p_1 t_1 + p_2 t_2)/\hbar} \sum_{\substack{d_1 \geq 0 \\ d_2 \geq 0}} e^{d_1 t_1 + d_2 t_2} \times$$

$$\times \frac{\prod_{-\infty}^{0} (p_2 - p_1 + m\hbar)^2}{\prod_1^{d_1} (p_1 + m\hbar)^2 \prod_1^{d_2} (p_2 + m\hbar) \prod_{-\infty}^{d_2 - d_1} (p_2 - p_1 + m\hbar)} =$$

$$= e^{(t_0 + p_1 t_1 + p_2 t_2)/\hbar} (1 + o(\frac{1}{\hbar})).$$

Corollary. $\vec{I} = \vec{J}$.

Therefore $\vec{I}_{(1)}$ is satisfied by the following system of differential equations:

$$\begin{cases} D_1^2 \vec{I}_{(1)} = q_1 (D_2 - D_1)^2 \vec{I}_{(1)} \\ D_2 (D_1 - D_2)^2 \vec{I}_{(1)} = q_2 \vec{I}_{(1)} \end{cases}$$

Where $D_i = \hbar \frac{d}{dt_i}, q_i = e^{t_i}, i = 1, 2$. Thus we get the relations in quantum cohomology of X_1:

$$\begin{cases} p_1^2 = q_1 (p_2 - p_1)^2 \\ p_2 (p_1 - p_2)^2 = q_2 \end{cases}$$

Example. 2. Let X_2 be a 3-dimensional toric manifold corresponding to the matrix

$$M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

X_2 is the projectivization of the sum of two trivial bunles and the square of the Hopf bundle. The image of the basis vectors under the projection $\mathbb{R}^5_+ \to \mathbb{R}^2$ used in the construction of X_2 is shown in Figure 2. It is clear that $c_1(T_{X_2}) = (0,3)$ is

FIGURE 2. Image of basis vectors of $\mathbb{R}^5_+$ in $\mathbb{R}^2$.

on the boundary of the Kahler cone (case 2). Using Kirwan's theorem we compute the cohomology:

$$H^*(X_2) = \mathbb{Z}[p_1, p_2]/(p_1^2, p_2(p_2 - 2p_1)^2).$$

We note that it is the same as $H^*(X_1)$ (recall that X_1 and X_2 are equivalent symplectic, but different complex manifolds). We compute the following formula:

$$\vec{I}_{(2)} = e^{(t_0+p_1t_1+p_2t_2)/\hbar} \sum_{\substack{d_1 \geq 0 \\ d_2 \geq 0}} e^{d_1t_1+d_2t_2} \times$$

$$\times \frac{\prod_{-\infty}^{0}(p_2-2p_1+m\hbar)^2}{\prod_1^{d_1}(p_1+m\hbar)^2\prod_1^{d_2}(p_2+m\hbar)^2\prod_{-\infty}^{d_2-2d_1}(p_2-2p_1+m\hbar)} =$$
$$= e^{(t_0+p_1t_1+p_2t_2)/\hbar}[1+\frac{2p_1-p_2}{\hbar}f(e_1^t)+o(\frac{1}{\hbar})],$$
where the function f is given by the formula

$$f(Q)=\sum_{d_1=1}^{\infty}\frac{(2d_1-1)!}{(d_1!)^2}Q_1^d.$$

To make $\vec{I}$ and $\vec{J}$ equal we have to perform some change of variables. We are lucky because f is quite simple, for example it satisfies the relation

$$Q\frac{d}{dQ}f=\frac{1}{\sqrt{1-4Q}}-1,$$

so the change of variable is easy to compute. Let Q_i (resp. q_i) , $i=1,2$, be the old variables (resp. new variables). The change of variables is given by the formulas $q_1=Q_1e^{2f(Q_1}$ and $q_2=Q_2e^{-f(Q_1)}$. The inverse change of variables is $Q_1=\frac{q_1}{(1+q_1)^2}, Q_2=q_2(1+q_1)$.

Let's denote $\tilde{D}_1=\hbar Q_i\frac{d}{dQ_i}$. $\vec{I}_{(2)}$is annihilated by the following system of differential operators

$$\begin{cases} \tilde{D}_1(\tilde{D}_2-2\tilde{D}_1)-Q_2 \\ \tilde{D}_1^2-Q_1(\tilde{D}_2-2\tilde{D}_1)(\tilde{D}_2-2\tilde{D}_1-\hbar) \end{cases}$$

Exercise. Check that under the described change of variables the system of equations for X_2 is transformed to the system of equations for X_1.

This is due to the fact that X_1 and X_2 are holomorphically different, but symplectically equivalent, and Gromov-Witten invariants are the invariants of symplectic structure.

2. **Idea for the proof of the Mirror Theorem.**

Let us move toward the proof of the Mirror theorem. So far we have no idea why $\vec{I}$ and $\vec{J}$ should be related at all. To understand "why?" let us forget everything and approach the problem of counting rational curves on manifolds in a straightforward way.

Problem: What is the number of rational curves of degree d on a quintic?

The direct way to approach the problem is to consider the maps

$$(x_1(w):\dots:x_5(w)):\mathbb{C}P^1\to\mathbb{C}P^4$$

of degree d with image on the quintic given by equation $F(x_1:\dots:x_5)=0$. Here w is a coordinate on $\mathbb{C}P^1$, $x_i(w)$'s are relatively prime polynomials of maximal degree d. Compactification of the set of such polynomials modulo $\mathbb{C}^*$ forms the space $X=\mathbb{C}P^{5d+4}$. Substituting into $F=0$ we get:

$$F(x(w))=a_0(x)+a_1(x)w+\dots+a_{5d}(x)w^{5d}=0.$$

Let $Y\subset X$ be the solution of the above equation. Y parametrizes the maps from $\mathbb{C}P^1$ to quintic.

Let $\mathcal{P}_{5d}$ be the trivial bundle over X with fiber $\mathbb{C}P^{5d+1}$. $F(x)$ is then a section of a bundle

$$\begin{array}{c} \mathcal{O}(5) \otimes \mathcal{P}_{5d} \\ \big\downarrow \\ \mathbb{C}P^{5d+4} \end{array}$$

and Y is the zero locus of this section. It follows that the dimension of Y is 3, which is the correct number. We have:

$$5 + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d} =< p \circ p, p >= \sum_{d=0}^{\infty} q^d \int_{\mathbb{C}P^{5d+4}} p^3 (5p)^{5d+1} =$$

$$= \sum_{d=0}^{\infty} q^d 5^{5d+1} = \frac{5}{1 - 5^5 q}.$$

Unfortunately this formula is incorrect. For example, it predicts the number of straight lines to be $5^6 \neq 2875 = 5^3 23$, which is known to be the correct number.

What's wrong? We didn't take into account that our cycle Y is invariant with respect to the Moebius transformation group on $\mathbb{C}P^1$.

Consider the action of the maximal torus S^1 of SL_2 on $\mathbb{C}P^1$, $\phi \in S^1 : w \to e^{\phi} w$. Computing the equivariant cohomology (which is the algebra over $H^*(BS^1) = \mathbb{Q}[\hbar]$ we get:

$$H^*_{S^1}(\mathbb{C}P^{5d+4}) = \mathbb{Q}[P, \hbar]/P^5 (P - \hbar)^5 \cdots (P - d\hbar)^5$$

Therefore the generating function for Gromov-Witten invariants is

$$\int_{[\mathbb{C}P^{5d+4}]} \mathrm{Euler}_{S^1}(\mathcal{O}(5) \otimes \mathcal{P}_{5d}) e^{Pz} = \frac{1}{2\pi i} \oint \frac{5P(5P - \hbar) \cdots (5P - 5d\hbar) e^{Pz} dP}{P^5 (P - \hbar)^5 \cdots (P - d\hbar)^5}.$$

This is of Duistermaat-Heckman formula type.

Let us recall that periods of integrals of holomorphic forms on mirrors of quintics are encoded in the formula

$$\vec{I}_{t,\hbar} = e^{pt} \frac{(5p + \hbar) \cdots (5p + 5d\hbar)}{(p + \hbar)^5 \cdots (p + d\hbar)^5}.$$

We see the resemblence with the formula above.

Claim.

$$\sum_{d=0}^{\infty} q^d \int_{[\mathbb{C}P^{5d+4}]} \mathrm{Euler}_{S^1}(\mathcal{O}(5) \otimes \mathcal{P}_{5d}) e^{Pz} =< \vec{I}_{t,\hbar}, \vec{I}_{\tau,-\hbar} >$$

under the change of variables $q = e^{\tau}$, $z = \frac{t - \tau}{\hbar}$. Here $<,>$ is the intersection index in $H^{2*}(X^3_{(5)})$.

Proof. . The proof is a straightforward computation. The key step is when we consider the integral near the pole $P = d'\hbar$ in the left hand side we should do the substitution $p = P - d'\hbar$.

We can emphasize two conclusions of this section:

1. We used simpler model, got the wrong number of rational curves on quintics, but we recovered the periods of mirror manifolds.

2.Let $q = e^{\tau}$, $qe^z \hbar = e^t$. Then $\vec{I} = e^{p ln q/\hbar}($ series in $q, 1/\hbar)$, but $< \vec{I}(qe^{z\hbar}, 1/\hbar), \vec{I}(q, 1/\hbar) >$ is a series in variables $q, z, \hbar$ and not $1/\hbar$. This is a polynomiality property of Duistermaat-Heckman formula.