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TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURES 21-22

A. GIVENTAL

In Lecture 21, we reviewed the supermanifold construction from the previous lecture, described an analogue for nonconvex bundles, and formulated a mirror theorem for toric complete intersections. In Lecture 22, we applied the mirror theorem in specific instances to obtain enumerative results consistent with what we already knew, and discussed a reformulation of the mirror theorem in terms of an integral that makes it look somewhat more like the original mirror conjecture.

1. Review of "supermanifolds"

Denote by V a vector bundle over the compact Kähler manifold X with total space E. In this section, we assume that V[1] is convex (generated by global sections).

In the previous lecture, we defined "relative" Gromov-Witten invariants of a complete intersection Y contained in X, that is, the zero locus of a section s of E transverse to the zero section. These differed from the ordinary Gromov-Witten invariants of Y in that they were defined on classes of X. To be precise:

$$A(T_1(c_1),\ldots,T_n(c_n))_{0,n,d}^{\Pi E} = \int_{[X_{0,n,d}]} \mathrm{ct}^*(A) \wedge_1^* (T_1)(c_1) \wedge \cdots \wedge_n^* (T_n)(c_n) \wedge (V_{0,n,d})$$

In this setting, we construct Gromov-Witten invariants in *nonzero* degrees as follows:

$$A(T_1(c_1), \dots, T_n(c_n))^E_{g,n,d}$$
$$= \int_{[X_{g,n,d}]} \mathrm{ct}^*(A) \wedge ({}^*_1 T_1)(c_1) \wedge \cdots \wedge ({}^*_n T_n)(c_n) \wedge {}_{S^1}(W_{g,n,d}).$$

Again, the action of S^1 is fibrewise on $W_{g,n,d}$.

An example where one uses this construction is the Calabi-Yau threefold, where we need to account for multiple covers. For a given generic[1] sphere, we can study the multiple covers of that sphere by working within its normal bundle. The normal bundle is generically $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which is precisely the situation described above.

The associated Frobenius structure in this setting lives on $H^*(X, (\lambda))$ with Poincaré pairing

$$\langle \phi, \psi \rangle = \int \phi \wedge \psi \wedge {}^{-1}_{S^1}(W)$$

and the diagonal form

$$\sum \eta^{\alpha\beta} \phi_\alpha \otimes \phi_\beta = (\text{Poincaré dual of diagonal}) \cdot {}_{S^1}(W).$$

Exercise. Derive the above formulae by modifying Kontsevich's modified WDVV construction for the convex case.

We note in passing that while the Gromov-Witten invariants for d nonzero are *a priori* integral, which was not the case in the convex case (because the formula for the Euler class of $V_{0,n,d}$ has a denominator), the Poincaré pairing is not integral-valued on the trivial class.

For grading purposes, the degree of the supermanifold ΠE is given by

$$c_1(\Pi E) = c_1(T_Y) = c_1(T_X) - c_1(V),$$

while in the concave case the appropriate term is

$$c_1(T_E) = c_1(T_X) + c_1(V).$$

3. The Mirror Theorem

We are now ready to state the mirror theorem for complete intersections in toric manifolds. That is, after we introduce some notation.

We first recall our standard construction of a toric manifold. Equip ^{n} with the standard action of the n-dimensional (real) torus T^n, and consider the moment map $\mu\colon {}^n \to Content - Length : 23772 Status : RO$

[1] I didn't quite catch what generic means here; the holomorphic spheres occur discretely, so we don't mean general position.

bbR$_+^n$. We then choose an integer matrix $M = (m_{ij})$ which maps $Content - Length: 23772 Status: RO$

bbR$_+^n$ down to $Content - Length: 23772 Status: RO$

bbR^r, or more invariantly, to the dual of the Lie algebra of the r-dimensional torus T^r. Another way of putting this is that M is the map induced on duals of Lie algebras by the embedding of T^r into T^n; this makes it clear that $M \circ \mu$ is the moment map of the torus action of T^r. Now choose a Kähler cone K (a maximal cone not containing the image of any of the basis vectors of $Content - Length: 23772 Status: RO$

bbR$_+^n$) and let X be the quotient of $(M \circ \mu)^{-1}(K)$ under the action of the complexified torus T^r.

We computed the T^n-equivariant cohomology of X previously:

$$H^*_{T^n}(X) = \frac{u_1, \dots, u_n, p_1, \dots, p_r, \lambda_1, \dots, \lambda_n]}{(u_j = \sum_i p_i m_{ij} - \lambda_j, \text{some monomials in } u_i)}.$$

To describe a complete intersection of codimension ℓ, we choose an integer matrix l_{ik} and consider the equivariant line bundles on X with first Chern classes $\sum_i p_i l_{ik}$ for $k = 1, \dots, \ell$. (Equivalently, we could have specified the torus characters induced by the line bundles.)

Let the torus T^ℓ act on the sum of the line bundles by (componentwise) scalar action on the fibres. Then the $G = (T^n \times T^\ell)$-equivariant cohomology of X is obtained from its T^n-equivariant cohomology by tensoring with $\lambda'_1, \dots, \lambda'_\ell)$; in this notation, the G-equivariant first Chern class of the k-th line bundle is

$$v_k = \sum_{i=1}^r p_i l_{ik} - \lambda'_k.$$

The mirror theorem will relate the H^*_G-valued generating function $\vec{J}$ (operators annihilating which yield relations in quantum cohomology) to another function $\vec{I}$ defined in terms of the combinatorial data (the matrices m_{ij} and l_{ik}).

We first introduce the analogue, in the relative setting, of the first row of the fundamental solution of the "Frobenius structure equation":

$$\vec{J} = e^{(t_0 + p_1 t_1 + \cdots + p_r t_r)/\hbar} \left[1 + \frac{1}{\hbar} \sum_{d \in \Lambda - \{0\}} e_*^{d_1 t_1 + \cdots + d_r t_r} \left(\frac{G(V'_{0,1,d})}{\hbar - c} \right) \right]$$

Notational reminders:

- Λ is the cone of degrees (first Chern classes) of holomorphic curves (or rather, of coordinates of such degrees with respect to the basis $t_1, \dots, t_r$ of $H^2(X)$);
- $: X_{0,1,d} \to X$ is evaluation at the marked point;
- c is the universal cotangent line at the marked point;
- $V'_{0,1,d}$ is the bundle over $X_{0,1,d}$ whose fibre at (Σ, f) is the set of global sections of $f^*(V)$ vanishing at the marked point (which differs topologically from the space of all global sections by a copy of V itself).

Now set $D_j(d) = \sum_i d_i m_{ij}$ and $L_k(d) = \sum_i d_i l_{ik}$, and define

$$\vec{I} = e^{(t_0 + \sum p_i t_i)/\hbar} \sum_{d \in \Lambda} e^{\sum d_i t_i} \frac{\prod_{m=1}^{L_k} (v_k + m\hbar)}{\prod_{m=1} D_j (u_j + m\hbar)}.$$

Beware that D_j can be negative (though not in many cases of interest); in this case, by $\prod_{m=1}^{D_j}(u_j+m\hbar)$ we actually mean $\prod_{m=D_j} -1(u_j+m\hbar)^{-1}$, or more suggestively,

$$\frac{\prod_{m=-\infty} 0(u_j+m\hbar)}{\prod_{m=-\infty} D_j(u_j+m\hbar)}.$$

No such trouble arises for L_k, which is always nonnegative.

Theorem 1. *Assume that $v_k \in \overline{K}$ for $k = 1, \ldots, r$, and that $\sum_j u_j - \sum_k v_k \in \overline{K}$. (That is, the line bundles are convex, and the first Chern class of the complete intersection is nonnegative.) Then $\vec{I}(t_0, t_i) = \vec{J}(t_0^{new}, t_i^{\text{new}})$ for a unique weighted homogeneous triangular change of variables:*

$$\begin{aligned} t_0^{\text{new}} &= t_0 + f_0(q)\hbar + \sum_j \lambda_j g_j(q) + \sum_j \lambda'_j g'_j(q) + h(q) \\ t_i^{\text{new}} &= t_i + f_i(q). \end{aligned}$$

In case you have as much trouble remembering term degrees as I do:

$$\deg t_i = 0, \deg t_0 = \deg \hbar = \deg \lambda_j = \deg \lambda'_j = 1, \deg q_i = \sum_j m_{ij} - \sum_k l_{ik}.$$

The uniqueness of the change of variables follows from writing out the first-order asymptotics of $\vec{I}$ and $\vec{J}$ in $1/\hbar$. Namely,

$$\vec{J} = e^{(t_0+pt)/\hbar}\left[1 + o\left(\frac{1}{\hbar}\right)\right],$$

while (calculation omitted!)

$$\vec{I} = e^{(t_0+pt)/\hbar} e^{f_0(q)}\left[1 + \frac{h(q)}{\hbar} + \sum \frac{\lambda_j g_j(q)}{\hbar} + \sum \frac{\lambda'_k g_k(q)}{\hbar} + \sum \frac{p_i f_i(q)}{\hbar} + o\left(\frac{1}{\hbar}\right)\right].$$

Explicitly determining the individual series f_i, g_j, g'_k, h can be tricky in practice, but one case is easy:

$$e^{f_0(q)} = \sum_{d \in \Lambda} \sum_{\sum L_k = \sum D_j} q^d \frac{\prod L_k!}{\prod D_j!}.$$

4. Some examples

We now apply the mirror theorem in some concrete examples, and verify that the results so obtained agree with what we already know. Specifically, let $X = P^{n-1}$ and let V be the line bundle $\mathcal{O}(\ell)$ over X, where $0 < \ell \leq n$. The bounds on ℓ ensure that V is convex, and that its first Chern class is nonnegative. We use the mirror theorem to deduce enumerative results about a hypersurface Y which is the zero locus of a section of V transverse to the zero section.

Let p denote the T^n-equivariant first Chern class of $\mathcal{O}(1)$; modulo the relation $(p - \lambda_1)\cdots(p - \lambda_n) = 0$, we have

$$\vec{I} = e^{(t_0+pt)/\hbar} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{m=1}^{\ell d}(\ell p - \lambda' + m\hbar)}{\prod_{j=1}^{n}\prod_{m=1}^{d}(p - \lambda_j + m\hbar)}.$$

Since we're interested in non-equivariant cohomology, we may as well set all of the λ_j, and λ', equal to 0. (Although this is the case in essentially all enumerative applications, actually proving the mirror theorem requires using equivariant cohomology.)

To apply the mirror theorem, we need to determine the change of coordinates taking $\vec{I}$ to $\vec{J}$. As described in the previous section, we do this by computing the first-order asymptotics of $\vec{I}$ in $1/\hbar$. Namely,

$$\vec{I} = e^{(t_0+pt)/\hbar} \sum_{d=0}^{\infty} e^{dt} \hbar^{(\ell-n)d} \left[\text{series in } \tfrac{1}{\hbar}\right].$$

The discussion now splits into three cases.

Case 1: $\ell < n-1$**.** In this case, the only term in $\vec{I}$ which is not $o(1/\hbar)$ is the $d=0$ term. Therefore no change of variables is needed: $\vec{I} = \vec{J}$. (This is true even without setting $\lambda = 0$.) In other words,

$$\vec{J} = e^{(t_0+pt)/\hbar} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{m=1}^{\ell d}(\ell p + m\hbar)}{\prod_{m=1}^{d}(p+m\hbar)^n}.$$

Since $\vec{J}$ is the first row of the fundamental solution of the usual differential equation, we'd like to say that the symbols of operators annihilating $\vec{J}$ give relations in the quantum cohomology of Y. However, there is a catch!

The catch is that the mirror theorem only refers to classes of Y pulled back from X, which means we are missing the middle cohomology classes. More precisely, the (ordinary) cohomology of Y decomposes into a direct sum of $p]/(p^{n-1})$ and its orthogonal complement. Fortunately, a vanishing theorem of G. Tian (based on the hard Lefschetz theorem) applies in this case,[2] to show that (ordinary) multiplication by p acts separately on $p]/(p^{n-1})$ and the complement, while multiplication by elements of the complement interchanges the two.

Therefore if we can write down a differential equation for $\vec{I}$, the relation in quantum cohomology we extract will be true (not just true modulo middle cohomology). In fact, we can write down such an equation.

$$\left(\hbar\frac{d}{dt}\right)^{n-1} \vec{I} = \ell e^t \left(\ell\hbar\frac{d}{dt} + \hbar\right) \cdots \left(\ell\hbar\frac{d}{dt} + (\ell-1)\hbar\right) \vec{I}. \tag{1}$$

[2]Tian's theorem probably would still apply if we were working in codimension greater than 1, but the discussion of this point ended inconclusively.

This is easier to verify, or even derive, than it looks. (It's even easier if, unlike me, you're familiar with hypergeometric functions.)

$$\begin{aligned}\left(\hbar\frac{d}{dt}\right)^{n-1}\vec{I} &= e^{(t_0+pt)/\hbar}\sum_d e^{dt}(p+d\hbar)^{n-1}\frac{\prod_{m=1}^{\ell d}(\ell p+m\hbar)}{\prod_{m=1}^{d}(p+m\hbar)^n}\\ &= e^t e^{(t_0+pt)/\hbar}\sum_d e^{dt}(p+(d+1)\hbar)^{n-1}\frac{\prod_{m=1}^{\ell(d+1)}(\ell p+m\hbar)}{\prod_{m=1}^{d+1}(p+m\hbar)^n}\\ &= e^t e^{(t_0+pt)\hbar}\sum_d e^{dt}\prod_{m=1}^{d+\ell-1}(\ell d+\ell d\hbar+m\hbar)\frac{\prod_{m=1}^{\ell d}(\ell p+m\hbar)}{\prod_{m=1}^{d}(p+m\hbar)^n}\\ &= \ell e^t\left(\ell\hbar\frac{d}{dt}+\hbar\right)\cdots\left(\ell\hbar\frac{d}{dt}+(\ell-1)\hbar\right)\vec{I}.\end{aligned}$$

Note that the factor $(\ell p+\ell(d+1)\hbar)$ cancels a factor of $p+(d+1)\hbar$ in the denominator, which saves us one differentiation on each side.

By replacing operators with their symbols in (4) and then setting $\hbar = 0$, we pull off the relation

$$p^{n-1} = \ell^\ell p^{\ell-1} q \tag{2}$$

in the quantum cohomology of Y. In applying this result, we must remember that the classes in (2) come from X and are indexed as such, which usually does not match the natural indexing on Y.

Example. Let $Y = P^1 \times P^1$ be a quadric surface in P^3, so that $\ell = 2$. Then $QH^*(Y)$ has the relations $p_1^2 = q_1, p_2^2 = q_2$ (see exercise below). The hyperplane section p pulls back to p_1+p_2 on Y (draw a plane meeting the quadric in two lines), and

$$(p_1+p_2)^3 = p_1q_1 + 3p_2q_1 + 3p_1q_2 + p_2q_2.$$

Also, straight lines on Y embed as straight lines in X, so when we pass to "relative" quantum cohomology, we must set $q = q_1 = q_2$. Thus we get the relation $p^3 = 4pq$, agreeing with (2). Note that since $H^2(X) \to H^2(Y)$ is not surjective, the mirror theorem doesn't even give us a full description of the small quantum cohomology of Y.

Exercise. Show that for arbitrary X_1 and X_2, we have an equality in small quantum cohomology

$$\mathrm{QH}^*(X_1\times X_2) = \mathrm{QH}^*(X_1)\otimes\mathrm{QH}^*(X_2).$$

Note that this relation does not generally hold in large quantum cohomology!

Example. Let $Y = G(4,2)$ be the Grassmannian of 2-planes in ^{4}, embedded as a quadric surface in P^5 (so again $\ell = 2$). Let c_1, c_2 and $\tilde{c_1}, \tilde{c_2}$ denote the Chern classes of the two tautological line bundles on Y. In Lecture 8, we computed that these four classes generate $\mathrm{QH}^*(Y)$ subject to the relations

$$(x^2+c_1x+c_2)(x^2+\tilde{c_1}x+\tilde{c_2}) = x^4+q,$$

whereas the mirror theorem predicts the relation $p^5 = 4qp$, where $p = -c_1 = \tilde{c_1}$. We may derive this as follows:

$$c_1^5 = c_1(c_1\tilde{c_1})^2 = c_1(-c_2-\tilde{c_2})^2 = (c_2+\tilde{c_2})(2c_1\tilde{c_2}) = 2c_1c_2\tilde{c_2} - 2\tilde{c_1}c_2\tilde{c_2} = 4c_1q.$$

Of course, the mirror theorem says nothing about the middle cohomology class c_2.

Case 2: $\ell = n-1$**.** In this case, the $d = 0$ and $d = 1$ terms of $\vec{I}$ will both contribute to the first-order asymptotics in $1/\hbar$:

$$\vec{I} = e^{(t_0+pt)/\hbar}\left(1 + e^t \frac{\prod_{m=1}^{\ell}(\ell p + m\hbar)}{\prod_{j=1}^{n}\prod_{m=1}^{1}(p + m\hbar)} + o\left(\frac{1}{\hbar}\right)\right)$$
$$= e^{(t_0+pt)/\hbar}\left(1 + \frac{e^t \ell!}{\hbar} + o\left(\frac{1}{\hbar}\right)\right)$$

Hence the change of variables in this case is $t_0^{\text{new}} = t_0 + e^t\ell!/\hbar$. That is,

$$\vec{J} = e^{-\ell! e^t/\hbar}\vec{I}.$$

This means that the differential equation satisfied by $\vec{J}$ should be precisely the equation obtained from by replacing $\hbar\frac{d}{dt}$ with $\hbar\frac{d}{dt} + \ell! e^t$. Consequently, the relation we get in quantum cohomology (plugging in $\ell = n-1$) is

(3) $$(p + (n-1)!q)^{n-1} = (n-1)^{n-1}q(p + (n-1)!q)^{n-2}.$$

Example. Let $Y^2_{(3)}$ denote a cubic surface in P^3 (also known as P^2 blown up at six points); the mirror theorem gives us the relation $(p+6q)^3 = 27(p+6q)^2$ in the (non-middle) quantum coholomogy. Since $\langle p \circ p, 1\rangle = \langle p, p\rangle = 3 + O(q)$ (the 3 is the number of times the intersection of two hyperplanes meets the cubic surface), we have

$$\langle p \circ p, p\rangle = \langle p^{\circ 3}, 1\rangle$$
$$= 9q\langle p, p\rangle + O(q^2) = 27q + O(q^2).$$

Hence the Gromov-Witten invariant $(p, p, p)_{0,3,1}$ equals 27; in other words, there are 27 lines on the cubic surface (since prescribing 3 marked points to lie at the intersections of a line with the three hyperplanes has no enumerative effect).

Example. Consider P^1 embedded as a conic in P^2. Then (3) gives the relation $(p + 2q)^2 = 4q(p + 2q)$, or $p^2 = 4q^2$ in quantum cohomology. This doesn't look right because p here is the pullback of the hyperplane section, which is twice the class of a point in P^1; likewise, the exponent of q counts degrees in multiples of the hyperplane section, so q represents what we would call q^2 in the quantum cohomology of P^1. Relabeling in terms of P^1 gives us the more familiar relation $p^2 = q$.

Case 3: $\ell = n$ **(Calabi-Yau).** As you may have guessed by now, this case is the messiest because the change of coordinates involves all of the terms in the series. We limit our attention to a single example.

Example. Let Y be a quintic hypersurface in P^4. Modulo the relation $p^4 = 0$,

$$\vec{I} = e^{(t_0+pt)/\hbar}\sum_{d=0}^{\infty} e^{dt}\frac{(5p+\hbar)\cdots(5p+5d\hbar)}{(p+\hbar)^5\cdots(p+d\hbar)^5}$$
$$= e^{(t_0+pt)/\hbar}\left[F_0(e^t) + \frac{p}{\hbar}F_1(e^t) + o\left(\frac{1}{\hbar}\right)\right],$$

where $F_0(q) = \sum_d q^d (5d)!/(d!)^5$. In this case, we make the substitutions

$$t_0^{\text{new}} = t_0 + \log(F_0(e^t)\hbar)$$
$$t^{\text{new}} = t + \frac{F_1(e^t)}{F_0(e^t)}$$

and conclude that $\vec{I} = \vec{J}(t_0^{\text{new}}, t^{\text{new}})$. This agrees with the prediction made in Lecture 9.

5. Mirror theorem on the wall, why so named by Givental?

Why do we call Theorem 1 a "mirror theorem"? In this section, we construct an oscillatory integral corresponding to $\vec{I}$ which makes the mirror theorem look somewhat more like the standard mirror conjecture. We will only obtain a mirror manifold of the right dimension in a special case.[3]

Consider the integral

$$I = \int_\Gamma e^{(\sum u_j - \sum v_k)/\hbar} u_1^{\lambda_1/\hbar} \cdots u_n^{\lambda_n/\hbar} v_1^{\lambda'_1/\hbar} \cdots v_\ell^{\lambda'_\ell/\hbar} \frac{\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n} \wedge dv_1 \wedge \cdots \wedge dv_\ell}{\frac{dq_1}{q_1} \wedge \cdots \wedge \frac{dq_r}{q_r}}$$

over a cycle Γ contained in the variety defined by the relations

$$\prod_{j=1}^n u_j^{m_{ij}} = q_i \prod_{k=1}^\ell v_k^{l_{ik}} \qquad i = 1, \ldots, r. \tag{4}$$

Define the differential operators

$$\partial_j = \sum_i m_{ij} \hbar q_i \frac{\partial}{\partial q_i} - \lambda_j, \qquad \partial'_k = \sum_i l_{ik} \hbar q_i \frac{\partial}{\partial q_i} - \lambda'_k$$

and write as before $D_j(d) = \sum_i d_i m_{ij}$ and $L_k(d) = \sum_i d_i l_{ik}$. Then

$$\Delta_d = \prod_j \prod_{m=0}^{D_j - 1} (\partial_j - m\hbar) - q^d \prod_k \prod_{m=1}^{L_k} (\partial'_k + m\hbar)$$

annihilates both the series $\vec{I}$ and the integral I, for every $d \in \Lambda$ such that D_j and L_k are nonnegative (otherwise the definition of Δ_d does not make sense). Although this looks like infinitely many differential equations, they all follow formally from some finite subcollection.

Exercise. Show that $\Delta_d I = 0$. It may help to substitute $u_i = e^{U_j}, v_k = e^{V_k}$; in these variables, applying Δ_d gives a form which is actually zero, not just exact.

This suggests that $\vec{I} = I$ for a suitable choice of the cycle Γ. In fact, since the integrand does not depend on the choice of the Kähler cone, different cycles could represent different toric manifolds.

While this equality looks somewhat like the mirror conjecture, the integration runs over too many variables. Let us see how to fix this (hereafter setting $\lambda = \lambda' = 0$) in the special case

$$l_{ik} = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \end{pmatrix}.$$

[3] My apologies for rendering this discussion somewhat sketchier than it was to begin with.

Let v_0 denote the sum of the first few u_i which have no 1 in their corresponding columns. Then v_1 is the sum of the next few u_i, v_2 the sum of a few after that, and so on. The integral reduces to

$$I = \int \frac{e^{v_0(u)/\hbar} \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}}{(1 - v_1(u)) \cdots (1 - v_\ell(u)) \frac{dq_1}{q_1} \wedge \cdots \wedge \frac{dq_r}{q_r}}.$$

The integrand has a first-order pole along the plane where the denominators are equal to 1, so we may use the residue theorem to reduce even further:

$$I = \int \frac{e^{v_0(u)/\hbar} \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}}{\frac{dq_1}{q_1} \wedge \cdots \wedge \frac{dq_r}{q_r} \wedge d(1 - v_1(u)) \wedge \cdots d(1 - v_\ell(u))},$$

where now the integration takes place over the variety (of dimension $n - l - \ell = \dim Y$, as desired) defined by (4) and

$$v_1(u) = \cdots = v_\ell(u) = 1.$$

The moral of the story (as best I could interpret it) is, to construct a mirror object, you should do something like take the spectrum of an algebra generated by certain cohomology classes.