TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 20

A. GIVENTAL

1. A Remark on Frobenius Structure

Recall the in Dubrovin's local classification of Frobenius structures (see lectures 15 and 16) near semi-simple points we have the following picture:

$$V(u) \in \mathfrak{so}_N^*, \quad H_i = \frac{1}{2} \sum_{i \neq j} \frac{V_{ij} V_{ji}}{u_i - u_j},$$
$$\{H_i, H_j\} = 0.$$

There is a nonlinear PDE associated with the classification :

$$\partial_i V = \{H_i, V\}$$

In the equivariant setting (for simplicity, $G = S^1$), $V(u, \lambda^{\pm}) \in \widehat{\mathfrak{so}}_N^*$, an affine Lie algebra. Consider

$$\mathcal{H}_i := \frac{1}{2\pi i} \oint H_i(V) \frac{d\lambda}{\lambda},$$

it satisfies similar equation $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$. A nonlinear problem which might be of interests to those working on integrable systems is :

$$\partial_i V = \{\mathcal{H}_i, V\}.$$

2. Equivariant Theory of Flag Manifolds

2.1. **Equivariant quantum cohomology of flag manifolds.** Let $X = F_n$ be the space of complete flags in $\mathbb{C}^n$. There is a natural $T^n \subset U_n$ action on X. Consider the fibre space

$$F_n \to BT^n \to BU_n.$$

The cohomological spectral sequence degenerates at E_2 and we have

$$H^*_{U_n}(F_n) = H^*(BT^n) = \mathbb{Q}[x_1, \dots, x_n].$$

Moreover, $H^*_{U_n}(F_n)$ is a free module of $H^*_{U_n}(\text{pt}) = \mathbb{Q}[c_1, \dots, c_n]$, where $c_i = \sigma_i(x_1, \dots, x_n)$, the elementary symmetric functions, and

$$H^*(F_n) = \frac{\mathbb{Q}[x]}{(\sigma(x))}.$$

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Notes taken by Y.-P. Lee.

I probably have changed some content of lecture according to my understanding. Please inform me of any errors.

Theorem 1.

$$QH^*_{U_n}(F_n) = \frac{\mathbb{Q}[x_1, \cdots, x_n; c_1, \ldots, c_n; q_1, \ldots, q_{n-1}]}{(c_i = \widetilde{\sigma}_i(x,q), \quad i = 1, \ldots n)}$$

where $\widetilde{\sigma}_i$ are the coefficients of the characteristic polynomial of the matrix:

$$\begin{pmatrix} x_1 & q_1 & 0 & \cdots & \cdots & 0 \\ -1 & x_2 & q_2 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -1 & x_{n-1} & q_{n-1} \\ \cdots & \cdots & \cdots & \cdots & -1 & x_n \end{pmatrix}$$

which are also the conservation laws of the Toda lattice.

The proof of this theorem is similar to that of nonequivariant case.

2.2. **Characteristic lagrangian variety.** In lecture fourteen we states in an exercise that (non-equivariant) quantum cohomology algebra is the ring of functions on a characteristic lagrangian variety. In the equivariant setting the equivariant quantum cohomology has a similar interpretation in the sense of families.

Consider the fibration $B = \{(c,t)\} \to SpecmH^*_{U_n}(\mathrm{pt}) = \{(c_1, \ldots, c_n)\}$, here t_i is defined by $q_i = e^{t_{i+1}-t_i}$. For fixed c's consider the cotangent space of the t-direction. These spaces form a complex space M fibered over B by $T^*_t\mathbb{C}^n$. M carries a natural Poisson structure $\sum \frac{\partial}{\partial t_i} \wedge \frac{\partial}{\partial x_i}$, and $SpecmQH^*_{U_n}(F_n) \subset M$ forms a family of lagrangian varieties over B. More precisely, a symplectic leaf is of the form $\{(x,t,c) \in M |\ c_i = \text{constant}, \quad i = 1, \ldots, n\}$ and $SpecmQH^*_{U_n}(F_n) \cap$ (one symplectic leaf) is a lagrangian variety

$$L = \{\widetilde{\sigma}_i(x,t) = c_i, \quad i = 1, \ldots, n\}.$$

We can also introduce $\vec{J}_{U_n}$, a vector dunction on B. The quantum differential equation reads:

$$D_i \vec{J}_{U_n} = c_i \vec{J}_{U_n},$$

where D_i are operators corresponding to quantum Toda lattice.

3. Equivariant Gromov-Witten Invariants on Convex Supermanifolds

So far we have discussed:

(1) Toric manifolds
(2) Equivariant cohomology
(3) Equivariant G-W invariants

Toward the end, we will formulate a mirror theorem for toric complete intersections, e. g. quintic hypersurface in $\mathbb{C}\mathbf{P}^4$. In order to prove the theorem, we need to extend this setting to the case of convex "super-manifolds".

3.1. Formulation.

Suppose $Y^{m-l} \overset{i}{\hookrightarrow} X^m$ is given by global sections of

$$V^l \to E \to X$$

such that $Y = s^{-1}(0)$. We are interested in

$$\int_{[Y]} i^*(\varphi) = \int_{[X]} \varphi \wedge Euler(E)$$
$$=: \int_{[\Pi E]} \varphi$$

where ΠE means the supermanifld whose underlying space is E with parity reversed in the fibre V^l direction.

We call V *convex* if it is spanned by global holomorphic sections. As we have seen in our previous exercise, this implies that for every holomorphic map f from genus 0 stable curve Σ to X

$$H^1(\Sigma, f^*V) = 0.$$

Therefore, $H^0(\Sigma, f^*V)$ forms a vector bundle $V_{0,n,d}$ over $X_{0,n,d}$:

$$\begin{array}{c} V_{0,n,d} \\ \Big\downarrow {\scriptstyle H^0(\Sigma, f^*V)} \\ X_{0,n,d} \end{array}$$

and $V_{0,n,d}|_f$ is independent of the choices of $\{f|[f] = [d]\}$. Also $f^*s \in H^0(\Sigma, f^*V)$ induces a section $s_{0,n,d}$ on $V_{0,n,d}$ whose zero locus is exactly $Y_{0,n,d}$. Thus

$$i_*[Y_{0,n,d}]^{vir} = [X_{0,n,d}]^{vir} \cap Euler(V_{0,n,d})$$

here the class $[d]$ in the subscript of Y is in $H := H^*(\Pi E)$ which is by definition $H^*(X)/ker(i^*)$, $i^* : H^*(X) \to H^*(Y)$. We can also introduce a Poicare pairing

$$< \varphi, \psi >= \int_{[X]} \varphi \wedge \psi \wedge Euler(V_{0,n,d})$$

which is nondegenerate on H. Can also define G-W invariants

$$\begin{aligned} & A(T_1(c_1), \ldots, T_n(c_n))^{X|Y} \\ := & \int_{[X_{0,n,d}]} \underline{ct}^*(A) ev_1^*(T_1)(c_1) \wedge \cdots \wedge ev_n^*(T_n)(c_n) \wedge Euler(V_{0,n,d}) \\ = & \sum_{d'_1+\ldots+d'_k=d} A(i^*(T_1), \ldots, i^*(T_n))^Y_{0,n,d} \end{aligned}$$

where $d \in H$ as always.

Remark. Perhaps this system of "restricted" G-W invariants on Y satisfies the general axioms of G-W theory.

3.2. **General properties.** (1) In the above formulation, we may use any multiplicative class instead of Euler class. For example the total chern class $\underline{chern} = \lambda^n + c_1(V)\lambda^{n-1} + \ldots + c_n(V)$ will do. However, it is equal to $Euler_{S^1}(V)$ for fibrewise $U(1)$-action.

If we use equivariant Euler class then $H^*_{S^1}(\Pi E)$ is equal to $H^*_{S^1}(X)$ over the field $\mathbb{C}(\lambda)$.

(2) *WDVV-argument* (Kontsevich's modification): A stable map f of degree d from a genus zero curve Σ with $4+n$ marked points to the target space X can be reinterpreted as two maps (f', f'') of degree (d', d'') from curves with $3+n'$ points and $3+n''$ points respectively to X with one constraint that the images of f' and f'' or their respective third point coincide.

Therefore we have the exact sequence

$$0 \to V_{0,4+n,d} \to V_{0,3+n',d'} \oplus V_{0,3+n'',d''} \xrightarrow{ev'_\times - ev''_\times} V \to 0.$$

The exactness follows from the convexity of V.

The Euler class is

$$Euler_{S^1}(V_{0,4+n,d} = \frac{Euler_{S^1}(V_{0,3+n',d'})Euler_{S^1}(V_{0,3+n'',d''})}{Euler_{S^1}(V)}$$

where V is the vector bundle on $X_{0,n,d}$ pulled back by $ev' = ev''$ of $V \to X$.

(3) We have a (non-conformal) $\mathbb{C}(\lambda)$-Frobenius structure on $H = H^*(X, \mathbb{C}(\lambda)) = H_{S^1}(\Pi E, \mathbb{C}(\lambda))$ with $< \varphi, \psi >= \int_{[X]} \varphi \wedge \psi \wedge Euler_{S^1}(V)$. But the G-W invariants are defined over $\mathbb{C}[\lambda]$ and gives non-equivariant invariants at $\lambda = 0$. We theorefore need some polynomiality properties.

(4) The same is true for G-equivariant G-W invariants ($G \times S^1$ action on E). The major advantage of this formulation is that we can use *fixed point formula*. Even though Y is not equivariant, the above formulation (replacing $[Y]$ by $[X] \cap Euler(V)$) enable us to use fixed point technique in this setting.