TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 19

A. GIVENTAL

1. Toric Manifolds and Equvariant Cohomology

We will state the results on the (equivariant) cohomology of toric manifolds. It's essentially F. Kirwan's theorem. Recall our commutative diagram of toric manifolds (we use the same notations as those in lectures 17 and 18):

$$\begin{array}{ccc} T^n : \mathbb{C}^n & \longrightarrow & \mathbb{R}^n_+ \\ \Big\uparrow & & \Big\downarrow {\scriptstyle M \text{ integer matrix}} \\ T^r & \longleftarrow & \mathbb{R}^r \end{array}$$

Then the toric manifold X is defined to be $J^{-1}(\omega)/T^r$, where ω is a point in K, an open cone in $\mathbb{R}^r$ (please refer to previous lectures). Assume that X is smooth, i. e. T^r action on $J^{-1}(\omega)$ is free, we have:

$$H^*(X) = H^*_{T^r}(J^{-1}(\omega))$$

We first notice that $J^{-1}(\omega)$ is T^n-equivariantly homotopic to $J^{-1}(K)$ where

$$J^{-1}(K) = \mathbb{C}^n \setminus \cup \text{ (Coordinate subspaces which miss } K \text{ under } J) \ .$$

and

$$H^*_{T^n}(\mathbb{C}^n, \mathbb{Q}) = \mathbb{Q}[u_1, \ldots, u_n] = H^*(BT^n),$$

$$H^*_{T^n}(J^{-1}(\omega)) = H^*_{T^n}(J^{-1}(K)) = \frac{\mathbb{Q}[u_1, \ldots, u_n]}{\mathcal{J}},$$

where $\mathcal{J}$ is the ideal of Σ. More precisely, $\mathcal{J}$ is the ideal generated by all monimials $u_{j_1} \cdots u_{j_s}$ such that $Me_{j_1}, \ldots, Me_{j_s}$ is a maximal subset whose convex hull does not intersects K. To go from T^n to T^r we use the following:

$$\begin{array}{ccc} ET^n/T^r & = & BT^r \\ \Big\downarrow {\scriptstyle T^n/T^r} & & \Big\downarrow {\scriptstyle T^n/T^r} \\ ET^n/T^n & = & BT^n. \end{array}$$

Then the spectral sequence associated with this fibration tells us how $H^*(BT^n)$ surjects $H^*(BT^r)$:

$$u_j \to \sum p_i m_{ij}.$$

Date: Oct. 28, 1997.

Notes taken by Y.-P. Lee.

The preparer of this note wants to apologies for his lack of competence to faithfully represents the content of the lecture.

Since

$$H^*(X) = H^*_{T^r}(J^{-1}(\omega)) = H^*_{T^r}(J^{-1}(K)),$$

we now have

Proposition 1.

$$H^*(X) = \frac{\mathbb{Q}[u_1, \ldots, u_n; p_1, \ldots, p_r]}{\mathcal{J} + I}$$

where I is the ideal of $\mathbb{Q}[u_1, \ldots, u_n; p_1, \ldots, p_r]$

$$I = (u_j - \sum_{i=1}^{r} p_i m_{ij}, \quad j = 1, \ldots, n).$$

If we use $T^n \times T^n$ action, we will find the equivariant cohomology of X:

Proposition 2.

$$H^*_{T^n}(X) = \frac{\mathbb{Q}[p_1, \ldots, p_r; u_1, \ldots, u_n; \lambda_1, \ldots, \lambda_n]}{\mathcal{J} + \widetilde{I}}$$

where

$$\widetilde{I} = (u_j - \sum_{i+1}^{r} p_i m_{ij} - \lambda_j, \quad j = 1, \ldots, n)$$

$$\mathbb{Q}[\lambda_1, \ldots, \lambda_n] = H^*_{T^n}(pt).$$

Once we have the equivariant cohomology of a toric manifold X, we may ask what is the Poincare pairing of equivariant cohomology classes. The key to the answer is the localization formula:

$$\begin{aligned} &\int_{[X]} \varphi_1 \varphi_2 \quad \in \mathbb{Q}[\lambda] \\ &= \sum_{j_1, \ldots, j_r} \mathrm{Res}_{u_{j_1} = \cdots = 0} \frac{\varphi_1(p,\lambda)\varphi_2(p,\lambda) dp_1 \wedge dp_r}{u_1(p,\lambda) \cdots u_n(p,\lambda)} \end{aligned}$$

which has limit as usual pairing when $\lambda \to 0$.

Remark. The (co)homology group of a toric manifold can also be computed using Morse theory: Choose a generic hamiltonian as Morse function whose critical manifolds are the even degree homology classes. One can also see that the cohomology classes are generated by second degree ones.

Example.

$$H^*_{T^n}(\mathbb{C}\mathbf{P}^n) = \frac{\mathbb{Q}[p, \lambda]}{(p - \lambda_1) \cdots (p - \lambda_n)}$$

where $u_j = p - \lambda_j$. The Poincare pairing is

$$\frac{1}{2\pi i} \oint \frac{\phi(p,\lambda) dp}{(p - \lambda_1) \cdots (p - \lambda_n)}.$$

2. Equivariant Gromov-Witten Invariants and Equivariant Quantum Cohomology

Let X be a compact kähler manifold and G a compact Lie group. Assume that G acts on X by (hamiltonian) automorphism of kähler structure. In this case G action lifts to $X_{g,n,d}$. In the case $g = 0$ and X convex $[X_{0,n,d}]$ can be used as equivariant fundamental class.

Remark. In general the construction of equivariant virtual fundanmental class is
1) addressed explicitly in Ruan's approach.
2) not addressed in Li-Tian or Behrend-Fantechi's approaches, but it seems possible to define $[X_{g,n,d}]_G^{vir}$ (folklore status). (c. f. Pandharipande-Graber's recent paper on localization formula.)

The diagram

$$\begin{array}{ccc}
X_{g,n+1,d} & & \\
\underline{ft}\big\downarrow & & \\
X_{g,n,d} & \xrightarrow{ev} & \underbrace{X \times X \cdots \times X}_{n} \\
\underline{ct}\big\downarrow & & \\
\overline{\mathcal{M}}_{g,n} & &
\end{array}$$

is G-equivariant. In particular, l_i's are also equivariant line bundles. Thus we can define the correlators

$$(T_1(c_1), \dots, T_n(c_n))_{g,n,d} \in H_G^*(\mathrm{pt})$$

where

$$T(c) = t^{(0)} + t^{(1)}c + t^{(2)}c^2 + \cdots .t^{(i)} \in H_G^*(X).$$

In one word, everything can be extended to equivariant setting.

Remark. Toric geometry somehow suggests that equivariant G-W invariants of X are related to non-equivariant G-W invariants of X/G. But there is no general statement at this point.

The relation of equivariant G-W invariants and enumerative geometry is more subtle. Let G be, say, S^1 and $a, b, c \in H_G^*(X)$, then $(a, b, c)_{0,3,d} \in \mathbb{Q}[\lambda] \cong H_G^*(\mathrm{pt})$. The constant term is nothing but the non-equivariant G-W invariant and hence has its (usual) enumerative meaning. But what is the enumerative meaning of the coefficient of λ^k?

Consider (the Poincare dual) of a, b, c as cycles in the homotopy quotient X_{S^1} which fibres over $\mathbb{C}\mathbf{P}^\infty = BS^1$ by X. The cell $\mathbb{C}\mathbf{P}^K \in \mathbb{C}\mathbf{P}^\infty$ induces an X fibre space $X_{S^1}^{(k)}$.

$$\begin{array}{ccc}
X_{S^1}^{(k)} & \longrightarrow & X_{S^1} \\
X\big\downarrow \pi^{(k)} & & X\big\downarrow \\
\mathbb{C}\mathbf{P}^k & \longrightarrow & \mathbb{C}\mathbf{P}^\infty
\end{array}$$

Concider the restriction of a, b, c to $H^*(X_G^{(k)})$ (still denote by the same letters). We now consider the numbers of curves in a fibre F of $\pi^{(k)}$ passing through the (restriction of) a, b, c.

This is exactly the enumerative meaning of the coefficent of λ^k. In other words, it is an intersection theory on E:

Now we are ready to write down the equivariant G-W potential. We begin with an example:

Example. 1. **Potential**: Consider the T^2 action on $\mathbb{C}\mathbf{P}^1$. The equivarint cohomology of $\mathbb{C}\mathbf{P}^1$ is spaned by $1, p, \lambda_1, \lambda_2$ with relation $(p-\lambda_1)(p-\lambda_2)=0$. It is a rank two free module of $\mathbb{C}[\lambda_1, \lambda_2]$. A general cohomology class (as $\mathbb{C}[\lambda]$-module) is written as $t_0 + t_1 p$. The potential can be written as

$$F(t_0, t_1, \lambda_1, \lambda_2, q) = \sum \frac{1}{n!} \sum_d q^d (t_0 + t_1 p, \ldots, t_0 + t_1 p)_{0,n,d}$$

First the degree of $(t_0, t_1, \lambda_1, \lambda_2, q)$ is $(1, 0, 1, 1, 2)$, and the degree of potential $F(t_0, t_1, \lambda_1, \lambda_2, q)$ is $3-1=2$.

The q^0 term is (n=3):

$$\begin{aligned}
F(t, q=0) &= \frac{1}{3!}(t,t,t)_{0,3,d} \\
&= (1/6)\int_{[\mathbb{C}\mathbf{P}^1]} (t_0 + t_1 p)^3 \\
&= (1/6)\oint \frac{(t_0^3 + 3t_0^2 t_1 p + 3 t_0 t_1^2 p^2 + t_1^3 p^3)dp}{(p-\lambda_1)(p-\lambda_2)} \\
&\text{substituting} \quad p^2 = (\lambda_1+\lambda_2)p - \lambda_1\lambda_2 \\
&= \frac{t_0^2 t_1}{2} + (\lambda_1+\lambda_2)\frac{t_0 t_1^2}{2}\frac{1}{6}[(\lambda_1+\lambda_2)^2 - \lambda_1\lambda_2]t_1^3.
\end{aligned}$$

The q^1 term should comes with the form $e^{t_1} q$ be divisor equation. But for degree reason it should be only of the form (constant)$e^{t_1} q$. Since λ has degree 2 it can not be in the constant. Therefore this constant must be the same as that of non-equivariant case, i. e. 1.

Since there can not be any q^2 or higher degree term, we have:

$$F(t_0, t_1, \lambda_1, \lambda_2, q) = \sum \frac{1}{n!} \sum_d q^d (t_0 + t_1 p, \ldots, t_0 + t_1 p)_{0,n,d} + e^{t_1} q.$$

2. **Equivariant quantum cohomology**:

$$QH^*_{T^2}(\mathbb{C}\mathbf{P}^1) = \frac{\mathbb{Q}[p, q, \lambda_1, \lambda_2]}{((p-\lambda_1)(p-\lambda_2) = q))}$$

3.$\mathcal{D}$**-module**:

$$(\hbar q \frac{d}{dq} - \lambda_1)(\hbar \frac{d}{dq} - \lambda_2)\vec{J} = q\vec{J}.$$

We can find (some computation needed here)

$$\vec{J}_{T^2} \equiv e^{p\ln q/\hbar} \sum_{d=0}^{\infty} \frac{q^d}{\prod_{i+1}^{2} \prod_{m=1}^{d} (p - \lambda_i + m\hbar)}$$
$$(mod \quad (p-\lambda_1)(p-\lambda_2) = 0).$$

4. **Frobenius structure**: Since T^2 acts on $\mathbb{C}\mathbf{P}^1$ by hamiltonian action, the equivariant quantum cohomology ring is a free $\mathbb{C}[\lambda]$-module. In fact, it is a free $\mathbb{C}[\lambda, q]]$-module with generators $1, p$. Consider the field of fractions $K = \mathbb{C}(\lambda, q))$, then there is a K-Frobenius structure on $H*$ (non-conformal). It's more convenient to view the Frobenius structure as that on a family:

There is an Euler vector field

$$E = \sum \deg(t) t \frac{\partial}{\partial t} + \sum (\text{constant}) q \frac{\partial}{\partial q} + \sum \lambda \frac{\partial}{\partial \lambda}$$

which, however, moves a fibre to another.