TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 18

A. GIVENTAL

Let G be a Lie group. Consider a topological space X on which G acts continuously. This lecture reveals the basic properties of the so-called equivariant cohomology $H_G^*(X)$ of X. It respects both the topology of the space X and the action of G. For more details the reader may consult Hsiang's book *Cohomology Theory of Topological Transformation Groups,* Audin's recent book *The Topology of Torus Action,* and Atiyah-Bott's paper *The Moment Map and Equivariant Cohomology* in Topology **23** (1984) 1–28.

1. Definition and basic properties of Equivariant Cohomology

As usually the universal G-bundle will be denoted by $EG \to BG$.

Definition 1. The equivariant cohomology of a G–space X is the usual cohomology of the space $X_G = (X \times EG)/G$:

$$H_G^*(X) = H^*(X_G).$$

Example. When the action of G on X is free X_G is a fiber bundle over X/G with simply connected fiber EG. Therefore X_G is homotopically equivalent to X/G and $H_G^*(X) = H^*(X_G) = H^*(X/G)$.

Example. If X is a point $X_G = EG/G = BG$ – the classifying space for the universal G-bundle EG (it is unique up to weak homotopy equivalence) and

$$H^*(pt) = H^*(BG).$$

Example. In the case of the simplest Lie group $G = S^1$, $EG = S^\infty, BG = \mathbb{CP}^\infty$ and $EG \to BG$ is the Hopf bundle $S^\infty \to \mathbb{CP}^\infty$.

$$H_S^*(pt) = H^*(BS^1) = \mathbb{Z}[\lambda] \quad \text{and} \quad \deg \lambda = 2.$$

More generally

$$H_{S^r}^*(pt) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots \lambda_r], \quad \deg \lambda_1 = \deg \lambda_2 = \cdots = \deg \lambda_r = 2.$$

Return to the general case. X_G is a bundle over BG. The fiber is X because the action of G on EG is free. So $H_G^*(X)$ is a natural $H_G^*(pt) = H^*(BG)$ module, i.e. $H^*(BG)$ is the coefficient ring for the theory. By the same reason a continuous equivariant map $\phi : X \to Y$ between two G spaces X and Y induces a pull back morphism $\phi^* : H_G^*(Y) \to H_G^*(X)$ of $H^*(BG)$ modules.

Exercise. Suppose that the action of G on X is free. Then $H^*(X/G) = H_G^*(X)$ will have a structure of a $H^*(BG)$ module. Describe this structure.

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Notes taken by M. Yakimov.

Hint In this case X is a G bundle over X/G and one can consider the characteristic classes of this bundle.

Corollary. The following inequality holds:

$$\mathrm{rk} H^*(X) \geq \mathrm{rk} H^*_G(X),$$

where $\mathrm{rk}H^*(X)$ is the rank of $H^*(X)$ considered as a $\mathbb{Q}$ vector space and $\mathrm{rk}H^*_G(X)$ is the rank of $H^*_G(X)$ considered as a vector space over the field of fractions of $H^*(BG)$ (i.e. localized).

Suppose that X has finite cohomological dimension (that is for any open subset U of X the cohomology groups $H^l(U)$ vanish for l large enough) and consider the case when G is a torus T^r. X^T will denote the fixed point set of the action. We have the equivariant embedding

$$i : X^T \to X,$$

which induces a homomorphism $i^* : H^*_T(X) \to H^*_T(X^T) = H^*(X^T) \otimes \mathbb{Q}[\lambda_1, \lambda_2, \cdots \lambda_n]$.

Theorem 1. *(Borel localization theorem) i^* is an isomorphism over $\mathbb{Q}(\lambda_1, \lambda_2, \cdots \lambda_n)$ (after localization), i.e. the kernel and the cokernel of i^* are torsion modules.*

Sketch of the proof $E_2^{p,q}$ for the Leray spectral sequence of $(X - X^T)_T \to (X - X^T)/T$ vanishes for $q \neq 0$ and one gets that $H^*_T((X - X^T)_T) = H^*((X - X^T)/T)$. So $H^*_T((X - X^T)_T)$ is a torsion module and the theorem follows from the exact T equivariant sequence of the pair (X, X^T).

Remark. The assumption that the space X is cohomologically finite dimensional is essential as shows the example with $X = EG$, $EG^G = \emptyset$, but $H^*_G(EG) = H^*(BG) \neq 0$.

As a consequence of the Borel localization theorem and the previous unequality we obtain:

Corollary. Smith's unequality. Under the same assumption

$$\mathrm{rk}_{\mathbb{Q}} H^*(X) \geq \mathrm{rk}_{\mathbb{Q}(\lambda_1, \lambda_2, \ldots \lambda_n)} H^*(X^T).$$

Exercise. Prove that the number of the connected components of a real curve of genus g is not greater than $g + 1$.

Hint Generalize the previous results for $\mathbb{Z}^2$ actions.

Similarly to the usual case one can associate to an equivariant map $f : N \to M$ between two compact G spaces N and M a push forward morphism

$$f_* : H^*(N) \to H^{* - \dim N + \dim M}(M).$$

(One considers finite dimensional approximations EG_n of EG and defines push forward morphism for the induced maps $f_n : X \times EG_n \to Y \times EG_n$, using Poincare pairing.) It also will be interpreted as integration along the fibers of the map f. The image of $\varphi \in H^*_G(X)$ under the map $X \to pt$ will be denoted by

$$\int_{[X]} \varphi \in H^*(BG).$$

Consider again a torus T acting on a space X. One proves that

$$i^* i_* \psi = \psi \wedge E_T(N), \text{ for } \psi \in H^*_T(X^T)$$

where N is the normal bundle to X^T in X and $E_T(N)$ is its equivariant Euler class. Equivalently:

$$i_* \frac{i^*\varphi}{E_T(N)} = \varphi \text{ for } \varphi \in H_T^*(X) \tag{1}$$

Example. If X^T is discrete (1) can be rewritten in a very explicit way. Suppose $x \in X^T$, then the action of T on X induces an action of T on T_xX (T fixes x). Every finite dimensional complex representation of T is a direct sum of one dimensional representations given by N characters $\chi_1(\lambda)$, $\chi_2(\lambda), \ldots, \chi_N(\lambda)$, and (1) reads:

$$\sum_{x\in X^T} \frac{(i^*\varphi)(\lambda)}{\chi_1(\lambda)\chi_2(\lambda)\ldots\chi_N(\lambda)} = \int_{[X]} \varphi,\ \varphi \in H_T^*(X).$$

In general $E_T(N) \in H^*(X^T) \otimes H^*(BT)$ has a nonzero term of degree 0 in the $H^*(X^T)$ grading. The remaining part of it is nilpotent and therefore it is invertible after localization by $\mathbb{C}[\lambda_1, \lambda_2, \ldots, \lambda_n] = H^*(BG, \mathbb{C})$. This explains the notation in (1).

2. **Duistermaat–Heckman formula**

Consider a Hamiltonian action of a torus T^r on a symplectic manifold (X, ω). The corresponding Hamiltonians are denoted by $H_1, \ldots, H_r$. The set of fixed points X^T is supposed to be finite. The principle term in the asymptotic expansion of the oscillating integral

$$\int_X e^{\lambda_1 H_1+\cdots+\lambda_r H_r} \frac{\omega^{\wedge N}}{n!} \quad \text{is} \quad \sum_{p\in X^T} \frac{e^{\sum \lambda_i H_i(p)}}{\sqrt{\text{Hessian}(\sum \lambda_i H_i)(p)}}.$$

The formula of Duistermaat–Heckman says that the latter is the precise value of this integral. Later this fact was nicely explained by Berline–Vergne and Atiyah–Bott in terms of the equivariant cohomology of X. They constructed a de Rham type complex with cohomology $H_T^*(X)$:

The space is $\Omega^*(X)^T[\lambda_1, \lambda_2, \ldots, \lambda_r]$ (T invariant forms with values in $\mathbb{R}[\lambda_1, \lambda_2, \ldots, \lambda_r]$.)

The differential is $D = d + \sum \lambda_j i_{v_{H_j}}$ ($\deg \lambda_i = 2$).

The form $p = \omega + \lambda_1 H_1 + \cdots + \lambda_r H_r$ is equivariantly closed (this is just another way of saying that v_{H_i} are vector fields with Hamiltonians H_i) and

$$\int_{[X]} e^p = \int_{[X]} e^{\sum \lambda_i H_i} \frac{\omega^{\wedge N}}{N!}$$

where the first integration is equivariant and the second is "usual". One gets the Duistermaat–Heckman formula applying to e^p the Borel localization formula (1).

Exercise. Consider a Hamiltonian action of S^1 on a symplectic manifold (X, ω) with a hamiltonian H. X^{S^1} is assumed to be finite. Prove that H is a perfect Morse function (*perfect* means that in the associated Morse complex the boundary operator should be trivial).

Hint Apply Smith and Morse inequalities.

The mirror partner of this statement is the following one: *If f is a holomorphic function on a complex manifold then $\Re f$ is a perfect Morse function.*

Corollary. In the above setting

$$\left(\int_{[X]} \varphi\right)\Big|_{\lambda=0} = \int_{[X]} \varphi|_{\lambda=0}$$

where in LHS the integration is equivariant and in RHS it is the usual one. Moreover the usual cohomology of X is "restriction" of the equivariant to $\lambda = 0$:

$$H^*(X) = H_T^*(X)/(\lambda_1, \dots, \lambda_r).$$

Proof The first statement follows from the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_T \\ \downarrow & & \downarrow \\ pt & \longrightarrow & BT \end{array}$$

In addition the first row gives a map $H_T^*(X) \to H^*(X)$. It descends to a map $H_T^*(X)/(\lambda_1, \dots, \lambda_r) \to H^*(X)$. The latter is an isomorphism and this can be seen from Borel's localization theorem.

Exercise. Consider the standard action of T^r on $\mathbb{C}^r$ defined by:

$$(e^{it_1}, \dots, e^{it_r})(x_1, \dots, x_r) = (e^{it_1}x_1, \dots, e^{it_r}x_r)$$

This action descends to an action of T^r on $\mathbb{CP}^{r-1}$. Then

$$H_{T^r}^*(\mathbb{CP}^{r-1}) = \mathbb{Z}[p, \lambda_1, \dots, \lambda_r]/(p - \lambda_1, \dots, p - \lambda_r),$$

where p is the equivariant first Chern class of the dual to the Hopf bundle. Prove also that for $\varphi \in H_T^*(\mathbb{CP}^{r-1})$ the equivariant integration on the fundamental cycle of $\mathbb{CP}^{r-1}$ is given by:

$$\int_{[\mathbb{CP}^{r-1}]} \varphi(p, \lambda) = \frac{1}{2\pi i} \int_{\text{circle around} \lambda_1, \dots \lambda_r} \frac{\varphi(p, \lambda) dp}{(p - \lambda_1) \dots (p - \lambda_r)}.$$