TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 17

A. GIVENTAL

The last time we discussed the mirror conjecture for $\mathbb{C}\mathbf{P}^4$ and quintics in $\mathbb{C}\mathbf{P}^4$. The formulas which we mentioned can be extended to the case of complete intersections in toric varieties.

1. Definitions

The basic definition is as follows.

Definition 1. An n-dimensional compact complex manifold X is called toric if it is equipped with an action of an n-dimensional complex torus $T^n_{\mathbb{C}}$ s. t. there exists a dense orbit isomorphic to $T^n_{\mathbb{C}}$.

Consider $\mathbb{C}^n$ with coordinates $x_1, x_2, \dots, x_n$ and the standard symplectic form $\Omega = -\frac{1}{2i}(\sum_{j=1}^n dx_j \wedge d\bar{x}_j)$ on it. There is a natural action of the real n-dimensional torus T^n on $\mathbb{C}^n$, namely the diagonal action

$$\operatorname{diag}(e^{it_1}, \dots, e^{it_n})(x_1, \dots, x_n) = (e^{it_1}x_1, \dots, e^{it_n}x_n).$$

Recall that:

Definition 2. A vector field v on a symlpectic manifold (M, Ω) is called Hamiltonian if there exists a function H s. t.:

$$\Omega + dH = 0.$$

H is called a Hamiltonian function of v. An action of a Lie group G on M is called Hamiltonian if the vector fields corresponding to the elements of Lie$G$ are Hamiltonian.

The above mentioned action of T^n on $\mathbb{C}^n$ is Hamiltonian and the associated moment map $\tilde{J} : \mathbb{C}^n \to \mathbb{R}^n_+ \subset \mathrm{Lie}^* T^n$ is given by:

$$\tilde{J}(x_1, \dots, x_n) = (|x_1|^2, \dots, |x_n|^2).$$

(We fix coordinates on $\mathrm{Lie}^* T$).

Consider a subtorus $T^r \subset T^n$. The moment map J for its action on $\mathbb{C}^n$ fits in the following commutative diagram:

$$\mathbb{C}^n \longrightarrow \mathbb{R}^n_+$$

where $\mathbb{R}^r_+ \subset \mathbb{R}^r = \mathrm{Lie}^* T^r$ and the map M is determined by the projection $\mathrm{Lie}^* T^n \to \mathrm{Lie}^* T^r$. A point $\omega \in \mathrm{Im} M \subset \mathbb{R}^r$ will be called generic if it is not situated on the image under M of any $r-1$-dimensional coordinate subspace in $\mathbb{R}^n$. For such an ω

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consider $J^{-1}(\omega) \subset \mathbb{C}^n$. It is invariant w.r.t. T^n and one can consider the quotient $X^{n-r}_{J,\omega} = J^{-1}(\omega)/T^r$. There is a natural action of $T^{n-r} = T^n/T^r$ on $X^{n-r}_{J,\omega}$.

Example. If $T \subset T^n$ is the diagonal embedding $e^{it} \mapsto \text{diag}(e^{it}, \ldots, e^{it})$, $t \in \mathbb{R}$ then

$$J(x_1, \ldots, x_n) = |x_1|^2 + \cdots + |x_n|^n,$$

ω is generic if $\omega \neq 0$ and $J^{-1}(\omega)/T^1 = \mathbb{C}\mathbf{P}^{n-1}$.

The form Ω is degenerated on $J^{-1}(\omega)$ (along the orbits of T^r) but on the quotient $X^{n-r}_{J,\omega}$ it is symplectic. The action of T^{n-r} on $X^{n-r}_{J,\omega}$ is Hamiltonian w.r.t. this form.

So far $X^{n-r}_{J,\omega}$ is only a real manifold. It is not clear at all that it can be equipped with a compatible complex structure. But let us consider the connected component of $\{\mathbb{R}^r \setminus$ images of $r-1$-dimensional coordinate subspaces in $\mathbb{R}^n\}$, containing ω. Denote it K and also let us denote

$$J^{-1}(K) = \mathbb{C}^n \setminus \cup\,(\text{Coordinate subspaces which miss } K \text{ under } J)$$

(here the inverse is only formal). Then $J^{-1}(K)$ is $T^n_{\mathbb{C}}$-invariant and the quotient $J^{-1}(K)/T^r_{\mathbb{C}}$ is isomorphic to $X^{n-r}_{J,\omega}$ (as real a manifold). Besides this obviously it comes equipped with an action of $T^{n-r}_{\mathbb{C}}$.

2. Main Properties

In this section we will collect some properties of the the toric manifolds $X^{n-r}_{J,\omega}$.

2.1 Compactness. The following theorem answers the question when $X^{n-r}_{J,\omega}$ is compact.

Theorem 1. *$X^{n-r}_{J,\omega}$ is compact iff one of the following 3 equivalent conditions holds:*

1.) $M^{-1}(0) = 0$

2.) $M^{-1}(\omega)$ *is compact*

3.) $J(\mathbb{C}^n)$ *fits into a half space in* $\mathbb{R}^r$

2.2 Smothness. In general $X^{n-r}_{J,\omega}$ is a nonsingular orbifold (it is nonsingular because ω does not lie on the images under M of the $r-1$ dimensional coordinate subspaces of $\mathbb{R}^n$ and therefore it is a regular value of J). Let the embedding of T^r in T^n be given by a matrix

$$M = (m_{ij})_{i=1,\ldots,r;j=1,\ldots,n}.$$

This is also the matrix of the projection M and that is why we use the same letter, hoping that it will not cause any confusion. Consider all r-dimensional faces $\mathbb{R}^I$ of $\mathbb{R}^n_+$, $I \subset \{1, \ldots, n\}$ $|I| = r$, whose image under M contains K. Each such face determines a point on $X^{n-r}_{J,\omega}$ (or rather on its complex version) whose stabilizer in $T^r_{\mathbb{C}}$ has order $|\det M^I|$. Here M^I is the minor associated with the subset $I \subset \{1, \ldots, n\}$. This implies the next theorem.

Theorem 2. *$X^{n-r}_{J,\omega}$ is a manifold, i.e. not an orbifold iff $|\det M^I| = 1$ for all minors M^I, $I \subset \{1, \ldots, n\}$ $|I| = r$, s.t. $M(\mathbb{R}^I_+)$ contains K.*

2.3 Minimal representation of a toric manifold. Here we will show that arbitrary Kähler toric manifold can be obtained by Hamiltonian reduction, described in Sect. 1. Suppose X^m is such a manifold, on which acts the complex torus $T^m_{\mathbb{C}}$. Averaging the Kähler form of X along the orbits of $T^m_{\mathbb{C}}$ one gets an invariant Kähler form on X.

It is not difficult to prove that the action of $T^m_{\mathbb{C}}$ is Hamiltonian w.r.t. it. Indeed suppose that H is a Hamiltonian function of a vector field corresponding to an element of Lie$T^m_{\mathbb{C}}$ which is multivalued along a closed curve. The complement in X of the open orbit of $T^m_{\mathbb{C}}$ isomorphic to $T^m_{\mathbb{C}}$ has a real codimension 2 and the curve can be deformed to lie entirely on this orbit. If we restrict the Kähler form of X to this orbit we will get a translationally invariant Kähler form on $T^m_{\mathbb{C}}$ for wihich the action of $T^m_{\mathbb{C}}$ on itself is not Hamiltonian, which is impossible.

The image under the associated moment map of X is an m-dimensional convex polyhidron (Atiyah, Bott). It can be embedded in $\mathbb{R}^n_+$ for sufficiently large n and for those n's X can be obtained by Hamiltonian reduction from $\mathbb{C}^n$. The minimal possible n gives a *minimal representation* of X.

2.4 $H^2(X^{n-r}_{J,\omega},\mathbb{Z})$ and the Kähler cone of $X^{n-r}_{J,\omega}$. Recall the construction of $X^{n-r}_{J,\omega} = J^{-1}(\omega)/T^r$. To each character of T^r one can associate a line bundle on $X^{n-r}_{J,\omega}$ glueing the trivial bundle over $J^{-1}(\omega)$ using this character. The Chern classes of these bundles can be identified with the the integer points in the standard lattice in $\mathbb{R}^r = \text{Lie}^*T^r$. So $H^2(X^{n-r}_{J,\omega},\mathbb{Z})$ coincides with this lattice and

$$H^2(X^{n-r}_{J,\omega},\mathbb{R}) = \text{Lie}^*T^r.$$

The Kähler cone of $X^{n-r}_{J,\omega}$ is precisely K.

2.5 $c_1(T_X)$. It is given by the formula

$$c_1(T_X) = Me_1 + \cdots + Me_n,$$

where the RHS is viewed as an element in $H^2(X^{n-r}_{J,\omega},\mathbb{Z})$ (cf. the previous subsection).

Consider $n-r$ coordinate vectors $e^*_{j_1},\ldots,e^*_{j_{n-r}}$ in Lie$T^n \cong (\mathbb{R}^n)^*$, that form a basis in LieT^n/LieT^r. They correspond to $n-r$ vector fields $\partial_1,\ldots,\partial_{n-r}$ on the toric variety $X^{n-r}_{J,\omega}$. The wedge product $\partial_1\wedge\cdots\wedge\partial_{n-r}$ is a section of the anticanonical bundle on $X^{n-r}_{J,\omega}$. $\partial_1,\ldots,\partial_{n-r}$ are linearly independent on an open subset of $X^{n-r}_{J,\omega}$ and the the pull back of the divisor where the above section is 0 on $J^{-1}(\omega)$ is a union of hyperplanes. The homology class of this divisor can be identified with the RHS of the formula for $c_1(T_X)$.

2.6 The cohomology algebra $H^*(X^{n-r}_{J,\omega},\mathbb{C})$. Here we will give a description of $H^*(X^{n-r}_{J,\omega},\mathbb{C})$ which is due to Kirwan. It is multiplicatively generated by $H^2(X^{n-r}_{J,\omega},\mathbb{C})$. First we need some notation. Denote by Σ the union of all $n-r$-dimensional coordinate subspaces in LieT^n which are orthogonal complements to those r-dimensional subspaces of Lie$^*T^n$ whose image under M contains K. By $\mathbb{C}[u_1,\ldots u_n]$ will be denoted the algebra of polynomial functions on the complecsification $(\text{Lie}T^n)_{\mathbb{C}}$ of LieT^n and by $\mathbb{C}[p_1,\ldots,p_r]$ the algebra of polynomials functions on $(\text{Lie}T^r)_{\mathbb{C}}$. Denote by I_Σ the ideal of Σ in $\mathbb{C}[u_1,\ldots u_n]$ and let I be the ideal in $\mathbb{C}[u_1,\ldots u_n,p_1,\ldots,p_r]$ generated by

$$u_j - \sum_{i=1}^{r} p_i m_{ij} \quad j = 1,\ldots n.$$

Then the formula of Kirwan says that

$$H^*(X^{n-r}_{J,\omega},\mathbb{C}) = \frac{\mathbb{C}[u_1,\dots u_n,p_1,\dots,p_r]}{I+I_\Sigma}$$

There is also a more explicit description of I_Σ. It coincide with the ideal $\mathcal{J}$ in $\mathbb{C}[u_1,\dots u_n]$ generated by all monomials $u_{j_1},\dots,u_{j_s}$ s.t. $Me_{j_1},\dots,Me_{j_s}$ is a maximal coordinate subset whose convex hull does not intersect K.

Finally one can see that the Kirwan formula respects the fact that we mentioned: that $H^*(X^{n-r}_{J,\omega},\mathbb{C})$ is generated by $H^2(X^{n-r}_{J,\omega},\mathbb{C})$. In view of Subsect. 2.5 $H^*(X^{n-r}_{J,\omega},\mathbb{C})$ should be a quotient of $\mathbb{C}[p_1,\dots,p_r]$ and that is the case because using the relations from I we can express u variables in terms of p variables and also the monomials generating $\mathcal{J}$ as polynomials in p.

The proofs of the facts in this Subsection will be given in Lect. 19.

2.7 Example. Consider two toric manifolds X_1 and X_2 which are projectvizations of the bundles $\mathcal{O}\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ and $\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(-2)$ respectivly over $\mathbb{C}\mathbf{P}^1$. The matrices of the minimal representation of these two toric varieties are

$$M_1 = \begin{array}{ccccc} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}$$

and

$$M_2 = \begin{array}{ccccc} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array}$$

where the first row comes from the induced action of T^1 on the basis $\mathbb{C}\mathbf{P}^1$ and the second from the diagonal action on the fiber. The projections of the coordinate vectors in $\mathbb{R}^5 = \mathrm{Lie}^*T^5$ on $\mathbb{R}^2$ and the cone K in the two cases are as follows:

(Here $e_1,\dots,e_5$ and f_1,f_2 are the coordinate vectors in $\mathbb{R}^5$ and $\mathbb{R}^2$ respectively.),i.e.

$$\text{Case 1:}\quad e_1\mapsto f_1, e_2\mapsto f_1, e_3\mapsto f_2, e_4\mapsto f_1-f_2, e_5\mapsto f_1-f_2,$$

$$\text{Case 2:}\quad e_1\mapsto f_1, e_2\mapsto f_1, e_3\mapsto f_2, e_4\mapsto f_1-2f_2, e_5\mapsto f_1-2f_2.$$

The cone K is exactly as shown because when we forget the second row of M this construction must give $\mathbb{C}\mathbf{P}^1$. One easely computes that I_1 is generated by the equations $u_1=u_2=p_1$, $u_3=p_2$, $u_4=u_5=p_2-p_1$ and I_2 is generated by $u_1=u_2=p_1$, $u_3=p_2$, $u_4=u_5=p_2-2p_1$. In both cases I_Σ is generated by u_1u_2 and $u_3u_4u_5$. Expressing the u variables in terms of the p variables using the above equations and applying the Kirwan's formula we get:

$$H^*(X_1,\mathbb{C}) = \mathbb{C}[p_1,p_2]/(p_1^2, p_2(p_2-p_1)^2$$

and

$$H^*(X_1,\mathbb{C}) = \mathbb{C}[p_1,p_2]/(p_1^2, p_2(p_2-2p_1)^2.$$

One can prove that X_1 and X_2 are symplecticly equivalent but not isomorphic as complex manifolds.