TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURES 15-16

A. GIVENTAL

Inspired by last lecture's dictionary between quantum cohomology and singularity theory, we may be tempted to ask if there is a Frobenius structure on the space Λ of miniversal deformations of our function f. This question was (essentially) first asked by K. Saito and answered by M. Saito:

Theorem 1. *There exists a volume form $\omega_\lambda = v(z,\lambda)dz_1 \wedge \cdots \wedge dz_m$ (called the "primitive form") such that the corresponding residue pairing*

$$< \phi, \psi >_\lambda = \frac{1}{(2\pi i)^m} \oint \frac{\phi(z)\psi(z)}{\frac{\partial f_\lambda}{\partial z_1} \cdots \frac{\partial f_\lambda}{\partial z_m}} \omega_\lambda$$

induces a flat metric on Λ.

The modern formulation of this is that the residue pairing $<,>_\lambda$ defined by the primitive form and the product induced by the identification $T_\lambda \Lambda = Q_\lambda$ form a Frobenius structure on Λ. This then begs the question - what is the counterpart of the connection[1] $\nabla_\hbar = \hbar d - \sum \phi_\alpha \circ dt_\alpha$ from quantum cohomology in singularity theory?

1. Oscillating Integrals

Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function, λ be a point in Λ, f_λ be the corresponding miniversal deformation, and ω_λ be a primitive form as supplied by Saito's theorem. Now consider the integral

$$I(\lambda) := \int_\Gamma e^{f_\lambda(z)/\hbar} \omega_\lambda.$$

Before we can make sense of this we need to define Γ, a real m-dimensional region in $\mathbb{C}^m$. Let $z \in \mathbb{C}^m$ be a non-degenerate critical point of f_λ and let $w \in \mathbb{C}$ be the corresponding critical value. Now choose a path γ in $\mathbb{C}$ which starts at w, which avoids all critical values thereafter, and whose real part approaches $-\infty$. Over γ and near w, the fibers of f_λ form a one-(real-)dimensional family of degenerating complex $(m-1)$-spheres (since z is a non-degenerate critical point) and inside this is a family of real $(m-1)$-spheres degenerating to z. The total space of this family is then m-dimensional and is what we call Γ. Note that although Γ is not compact, the integral $I(\lambda)$ converges since the integrand decreases rapidly enough at infinity. Of course the value of the integral depends on the choices of γ and Γ, but only through the homology class of Γ. This will come up again later.

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Notes taken by Jim Borger.

[1] $\nabla_\hbar$ is obviously not really a connection but a differential operator which is $\hbar$ times a connection.

This brings us to the counterpart of $\nabla_\hbar$ in singularity theory. Let $L = \mathrm{Spec}(T_H)$ (where the algebra structure on T_H is that given by the Frobenius structure). The algebra map $S^{\cdot}(T_H) \to T_H$ realizes L as a Lagrangian subvariety of the cotangent bundle T^*H. Now we can think of a miniversal deformation f_λ as a function on the cotangent space at λ. Under this identification, the critical points of f_λ are the points of the fiber L_λ of L over λ.

Now fix $\lambda \in \Lambda$, let $\{\phi_\alpha\}$ be a basis of $Q_\lambda = T_\lambda H$, and let $\{\Gamma_\beta\}$ be the cycles around the points of L_λ constructed above[2]. Then define the functions

$$S_{\alpha\beta}(\lambda, x) = \int_{\Gamma_\beta} e^{f_\lambda(z)/\hbar} \phi_\alpha(x, \lambda)\omega_\lambda$$

on L_λ, i.e. in Q_λ. A consequence of Saito's work (but not of the theorem quoted above) is that the $S_{\alpha\beta}$ form a fundamental solution set to the differential equation $\hbar\partial_i S = \phi_i \circ S$ (where the ∂_i denote differentiation with respect to the flat coordinates on Λ guaranteed by the theorem quoted above).[3] In particular, the "first row" of $(S_{\alpha\beta})$ is

$$I = \int_{\Gamma_\beta} e^{f_\lambda(z)/\hbar} \omega_\lambda.$$

Exercise. Show that

$$\int_{-1}^{1} (\alpha + \beta x + \cdots) e^{-ax^2 - bx^3 - \cdots} dx$$

is asymptotically $\sqrt{\frac{\hbar}{2\pi a}}(\alpha + O(\hbar))$.

Theorem 2. *(Dubrovin) Suppose H is a manifold with conformal Frobenius structure and λ is a semi-simple point of H. Let $u_1, ..., u_N$ be the branches near λ of the critical value function and let U be $diag(u_1, ..., u_N)$. Then in a neighborhood of λ, the fundamental solution S of $\nabla_\hbar S = 0$ can be written in the form*

$$S = \psi(1 + \hbar R + \hbar^2 R^{(2)} + \hbar^3 R^{(3)} + \cdots)e^{U/\hbar}.$$

where ψ, R, and $R^{(*)}$ are $n \times n$ matrices whose entries are germs at λ of sections of the Frobenius algebra T_H.

Remark. The explicit form given by the theorem is visibly analogous to the expansion

$$S_{\alpha\beta} := \int_{\Gamma_\beta^m} e^{f_\lambda(z)/\hbar} \phi dz = \hbar^{m/2} e^{f_\lambda(0)/\hbar} \frac{\phi(a)}{\Delta(a)^{1/2}}(1 + o(1)).$$

in singularity theory.

The proof of this theorem proceeds by finding a list differential equations the unknown matrices must satisfy and then showing certain integrability conditions are satisfied. We will now analyze some of these differential equations.

Restricting to a neighborhood of the semi-simple point λ, we can assume that the Frobenius structure on the entire manifold is semi-simple, i.e. the algebra T_H is the product of n copies of the function algebra of H. Said another way, the Lagrangian submanifold L of T^*H is the disjoint union of n copies of h. Since

[2] It appears that these should be chosen in a coherent manner, so this can only be done generically and locally

[3] It is easy to see using the stationary phase approximation that the S are solutions to the first approximation in $\hbar$.

the pairing $<,>$ is non-degenerate, we can normalize the characteristic functions of the connected components to produce a list of functions $v_1, ..., v_n$ (on L) which is orthonormal with respect to $<,>$, i.e. $< v_i, v_j >= \delta_{ij}$. In particular, the v_i form a simultaneous eigenbasis for the multiplication maps $v\circ$.

Now, the usual presentation $\nabla_\hbar = \hbar d + A^1$ is in $<,>$-flat coordinates, but it will be more useful to work with coordinates in which A^1 is diagonalizable, so let ψ be the transition matrix from the $<,>$-flat basis of T_H to the orthonormal basis of v's. Then $\psi^{-1}A^1\psi = -dU$ where U is the diagonal matrix whose entries are $u_1, ..., u_n$, the branches of the potential function restricted to L. The u's form a coordinate system and are what Dubrovin calls the "canonical coordinates".

We can now deduce some of the necessary conditions alluded to above by substituting the explicit representation given in the theorem into the differential equation $\hbar dS = A \wedge S$. Considering the $\hbar^0$ terms, we see immediately that $A \wedge \psi = \psi dU$. From the $\hbar^1$ terms, we get

$$d\psi + \psi R \cdot dU = A \wedge \psi R = \psi \cdot dU \cdot R \qquad \text{so} \qquad d\psi = \psi[R, dU].$$

This differential equation is under-determined, but we can change this. Let D be the differential operator $d + \psi(d\psi)$ and let E be the Euler field given by the conformal structure. Then $D^2 = 0$ and $L_E R = -R$ (since the degree of R is -1). These two are the additional constraints which make the system well-determined, and they can put into a familiar form.

Define $V \in \mathfrak{so}_N$ by $V_{ij} = (u_i - u_j)R_{ij}$ and

$$H_i = \frac{1}{2}\sum_{i \neq j} \frac{V_{ij}V_{ji}}{u_i - u_j}.$$

Theorem 3. *(Dubrovin) For all i, j, m, n, $\{H_i, H_j\} = 0$ and $\frac{\partial}{\partial u_i} V_{mn} = \{H_i, V_{mn}\}$*

Now we briefly return to quantum cohomology.

Conjecture 1. *(Givental) Let X be a compact Kähler manifold and write H for $H^*(X)$. Define the function $G : H^*(X) \to \mathbb{C}$ by*

$$G(t) = \sum_{n=0}^{\infty} \sum_{d \in \Lambda} (t, ..., t)_{1,n,d}.$$

If the Frobenius structure on H is generically semi-simple (for example flag manifolds), then

$$dG = \frac{1}{48}\sum_{i=1} N \frac{d\Delta_i}{\Delta_i} + \frac{1}{2}\sum_{i=1}^{N} R_{ii} du_i$$

at semi-simple points. Here the Δ_i are the Hessians at the critical points defined in lecture 14, and the R_{ii} are the diagonal entries of the matrix R given by Dubrovin's theorem.

Exercise. Check this conjecture for $\mathbb{C}\mathbf{P}^1$.

2. **The Mirror Conjecture**

The mirror conjecture loosely says that the Gromov-Witten invariants of a manifold X should correspond to oscillating integrals of singularity theory type on another manifold Y, called the "mirror partner" of X.

Examples.

(1) $X = \mathbb{C}\mathbf{P}^{m-1}$. We know from lecture 8 that

$$QH^*(X) = \mathbb{Q}[p,q]/(p^{\circ m} - q)$$

and from lecture 10 that the differential operator $(hq\frac{d}{dq})^m - q$ annihilates $\vec{J}$, the "first row" of the matrix S of fundamental solutions to the differential operator $\nabla_\hbar$. Now for a complex variable q, define the integral

$$I(q) = \int_{\Gamma \subset (z_1 \cdots z_m = q)} e^{(z_1+\cdots+z_m)/\hbar} \frac{dz_1 \wedge \cdots \wedge dz_m}{dq}.$$

As described above, Γ is an appropriately chosen non-compact $(m-1)-$cycle Γ in the subvariety $z_1 \cdots z_m = q$ of $\mathbb{C}^m$. Each possible Γ is based at some critical point of $(z_1 + \cdots + z_m)$ restricted to the subvariety $z_1 \cdots z_m = q$, and it is easy enough to find these using Lagrange multipliers. (Let $\log(z_1) + \cdots + \log(z_m) - \log(q)$ be the constraint function and p denote the multiplier. Then $1 = p/z_i$, so $p^m = q$.) It turns out that I satisfies the same differential equation that $\vec{J}$ does. (Exercise.) In fact, for each of the m critical points, we get a Γ, hence an I, and this list of functions I makes up the entries of $\vec{J}$.

(2) X is a quintic threefold in $\mathbb{C}\mathbf{P}^4$. In lecture 9, we had the notation

$$K(q) = 5 + \sum_{n=0}^{\infty} n_d \frac{d^3 q^d}{1 - q^d},$$

where the (n_d) are determined by some differential equations, one of which is

$$D^4 I = 5q(5D+1)(5D+2)(5D+3)(5D+4)I,$$

where $D = q\frac{d}{dq}$. The classical mirror conjecture from physics states that these (n_d) are in fact the same (n_d) determined by Gromov-Witten theory (see the very end of lecture 8).

Now let X_λ be the family of quintic threefolds in $\mathbb{C}\mathbf{P}^4$ defined by the homogeneous equation $X_1 \cdots X_5 = \lambda^{1/5}(X_1^5 + \cdots X_5^5)$. (Assume λ varies over a region where we can choose a branch of the fifth root function.) Since X_λ is a projective hypersurface, the Lefschetz theorems tell us the off-diagonal entries on the Hodge diamond. Since it's Calabi-Yau, we know $h^{3,0} = h^{0,3} = 1$. Finally, $h^{2,1} = h^{1,2} = 126 - 25 = 101$, the number of quintic coefficients minus the dimension of projective transformations. Therefore the complete Hodge diamond is

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 101 & 1 & 0 \\ 0 & 1 & 101 & 0 \\ 1 & 0 & 0 & 1 \end{array}.$$

Now $G = (\mathbb{Z}/5)^4$ acts on X_λ by multiplication by roots of unity. There is a Calabi-Yau desingularization Y_λ of X_λ/G with Hodge diamond

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 101 & 0 \\ 0 & 101 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}.$$

If we let $\omega_\lambda^{3,0}$ be the volume form on Y_λ, then each solution I to the above differential equation satisfies

$$I(\lambda) = \int_{\partial^3} \omega_\lambda^{3,0}$$

for appropriate choice of a 3-cycle ∂^3.

This brings us to a more general version of the mirror conjecture: Let X^m be a compact Kähler manifold. Then there should be a family Y_λ^m of manifolds with holomorphic volume forms $\omega_\lambda^{m,0}$ and global functions f_λ such that the entries of the $\vec{J}$ from Gromov-Witten theory are the functions

$$I(\lambda) = \int_{\Gamma \subset Y_\lambda} e^{f_\lambda(z)/\hbar} \omega_\lambda^{m,0}$$

given by the various choices of Γ.

In the case X is Calabi-Yau, we expect in addition that the Y_λ are compact Calabi-Yau manifolds. Then $\omega_\lambda^{m,0}$ is just a multiple of the volume form and f_λ is constant, so (up to multiplication by constants) the volume form and function don't give us any additional information. For those interested only in Calabi-Yau manifolds, the only important information is then the family of manifolds itself, and this why the extra data of the volume form and complex-valued function, which are necessary in the general case, were not seen by physicists.

(3) Now what about functoriality of the mirror process? Especially, in our motivating example of quintic threefolds, how does the mirror relate to the mirror of the ambient projective space? The mirror of $\mathbb{C}\mathbf{P}^4$ is the hypersurface of $\mathbb{C}^5$ defined by the equation $z_1 \cdots z_5 = q$ with function $f(z) = z_1 + \cdots + z_5$ and volume form $\omega = \frac{dz_1 \wedge \cdots \wedge dz_5}{dq}$. We can rewrite I (for $\mathbb{C}\mathbf{P}^4$) by first integrating along the fibers of f:

$$I = \int_{c=\text{crit.val.}}^{-\infty} e^{c/\hbar} dc \int_{Y_q, f(z)=c} \frac{dz_1 \wedge \cdots \wedge dz_5}{d(z_1 \cdots z_5) \wedge d(z_1 + \cdots + z_5)}.$$

If we now denote the inner integral as $I_X(q)$, then it satisfies the differential equation

$$D^4 I_X = 5q(5D+1)(5D+2)(5D+3)(5D+4) I_X$$

so we might hope that it's closely related to $I(\lambda)$ of the previous example.

We can rewrite the defining equation of X_λ as

$$\frac{\lambda^{1/5} x_1^5}{x_1 \cdots x_5} + \cdots + \frac{\lambda^{1/5} x_5^5}{x_1 \cdots x_5} = 1.$$

So setting $z_i = \lambda^{1/5} x_i^5 / x_1 \cdots x_5$, we see that an affine open piece of X_λ is described as a subvariety of $\mathbb{C}^5$ by the equations $z_1 + \cdots + z_5 = 1$ and $z_1 \cdots z_5 = \lambda$. Then

$$\frac{dz_1 \wedge \cdots \wedge dz_5}{d(z_1 \cdots z_5) \wedge d(z_1 + \cdots + z_5)}.$$

is a holomorphic volume form on X_λ, and it descends to volume form on X_λ / G which can then be pulled back to a volume form on Y_λ. It therefore agrees with $\omega_\lambda^{3,0}$ up to a constant multiple, and this implies that I and I_X also agree up to a constant multiple.