TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 14

A. GIVENTAL

1. Frobenius structures

Let us now go back to studying quantum cohomology in general. We shall introduce an axiomatic approach, due to Dubrovin, to studying Gromov-Witten invariants by way of Frobenius structures, which will provide a geometric way of thinking about the WDVV equation, allow us to explore analogies between GW theory and singularity theory, leading to the ideas of mirror symmetry.

Definition 1. A *Frobenius algebra* is a commutative associative algebra A with unit $\mathbf{1}$ equipped with a linear functional $\alpha : A \to \mathbb{C}$ such that the pairing $< a, b >:= \alpha(ab)$ is nondegenerate.

Example. (not used in the sequel) The representation ring of a finite group.

Definition 2. A*Frobenius structure* on a manifold H is a Frobenius structure on each tangent space $T_t H$ such that

1. The metric $<,>$ is flat ($\nabla^2 = 0$)
2. The vector field $\mathbf{1}$ is covariantly constant ($\nabla \mathbf{1} = 0$)
3. The system of PDE's
$$\hbar \nabla_w s = w \circ s$$
is integrable $\forall \hbar \neq 0$, where w and s are vector fields and $\circ$ denotes the Frobenius multiplication. In ∇-flat coordinates $\{t^\alpha\}$, this means that the family of connections $\nabla_\hbar := \hbar d - A_\alpha(t) dt^\alpha \wedge$ is flat for all $\hbar \neq 0$, where $A_\alpha = \partial_\alpha \circ$.

A Frobenius structure is called *conformal of dimension D* if H is equipped with a vector field E(uler) such that $\mathbf{1}$, $\circ$ and $<,>$ are eigenvectors of the Lie derivative operator L_E with eigenvalues -1, 1 and $2 - D$, respectively.

Example. For X compact Kähler, let $H = H^*(X)/2\pi i H^2(X, \mathbb{Z})$. Then H is a Frobenius manifold of conformal dimension $D = \dim_{\mathbb{C}} X$. Here $\circ$ is the quantum cup product, $<,>$ is the Poincaré pairing, α - integration over the fundamental cycle, $\mathbf{1}$ the fundamental class, and

$$E = t^0 \frac{\partial}{\partial t^0} + \sum_{i=1}^{r} c_i \frac{\partial}{\partial t^i} + \sum_{\alpha : \deg t^\alpha < 0} (\deg t^\alpha) t^\alpha \frac{\partial}{\partial t^\alpha},$$

where t^0 is the coordinate on H^0, t^i on H^2, t^α on $H^{>2}$, c_i the components of $c_1(TX)$, and $\deg t^\alpha = 1 - \frac{1}{2} \deg \phi_\alpha$.

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Notes taken by Dmitry Roytenberg.

There exists a clasification of semi-simple Frobenius structures due to Dubrovin.

Let us now examine analogies between GW theory and singularity theory in the framework of Frobenius structures.

(0) In GW theory, the basic object is an almost Kähler manifold X; in singularity theory it is $f : \mathbb{C}^m_{,0} \to \mathbb{C}_{,0}$ – a germ of a holomorphic function at an isolated critical point. Normal forms of simple singularities correspond to simple Lie algebras of types A, D and E:

$$\begin{array}{rcl} A_\mu & \leftrightarrow & x_1^{\mu+1} + \sum_{i=2}^{m} x_i^2 \\ D_\mu & \leftrightarrow & x^2 y - y^{\mu-1} \\ E_6 & \leftrightarrow & x^3 + y^4 \\ E_7 & \leftrightarrow & x^3 + xy^3 \\ E_8 & \leftrightarrow & x^3 + y^5 \end{array}$$

(1) The analogue of $H^*(X)$ in singularity theory is the *local algebra of a critical point*:

$$Q = \mathbb{C}[[z]]/(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_m})$$

Example. For singularity of type A_2 with $m = 1$, $f(x) = x^3$, and $Q = \mathbb{C}[[x]]/(x^2) =< 1, x >$, $x^2 = 0$.

(2) Corresponding to the Poincaré pairing on H^*, in singularity theory one has the *residue pairing*:

$$\langle \phi, \psi \rangle = \frac{1}{(2\pi i)^m} \oint_{\left|\frac{\partial f}{\partial z_j}\right| = \epsilon_j} \frac{\phi(z)\psi(z) dz_1 \cdots dz_m}{\frac{\partial f}{\partial z_1} \cdots \frac{\partial f}{\partial z_m}}$$

Example. For A_2, the pairing is

$$\langle \phi, \psi \rangle = \frac{1}{2\pi i} \oint \frac{\phi(x)\psi(x) dx}{3x^2}$$

so $\langle 1, 1 \rangle = \langle x, x \rangle = 0$, $\langle 1, x \rangle \neq 0$.

(3) The Frobenius manifold in GW theory is $H = H^*(X)/2\pi i H^2(X, \mathbb{Z})$; in singularity theory it is the *base of miniversal deformation* Λ: in the space of germs of holomorhic functions, we pick a transverse slice to the orbit of f under local diffeomorphisms; Λ is the paramater space for this slice. In other words, a miniversal deformation of f is

$$f(z, \lambda) = f(z) + \lambda_1 \phi_1(z) + \cdots + \lambda_\mu \phi_\mu(z),$$

where $\{\phi_i\} = \{ \frac{\partial f}{\partial \lambda_i}\Big|_{\lambda=0} \}$ is a basis of the local algebra Q. Thus $\dim \Lambda = \dim Q$.

Example. For A_2, $f(z, \lambda) = f(z) + \lambda_1 x + \lambda_0$

(4) The singularity-theory analogue of quantum cup product $\circ : T_t H \times T_t H \to T_t H$ is the multiplication in the algebra of functions on the critical set of $f_\lambda(z) = f(z, \lambda)$: $Q_\lambda = \mathbb{C}[z]/(\frac{\partial f_\lambda}{\partial z})$. We map $T_\lambda \Lambda$ to Q_λ by $\frac{\partial}{\partial \lambda_i} \mapsto \frac{\partial f_\lambda}{\partial \lambda_i}$. This turns out to be an isomorphism.

Example. In A_2 case, $\frac{\partial}{\partial \lambda_0} \mapsto 1$, $\frac{\partial}{\partial \lambda_1} \mapsto x$, and so $\frac{\partial}{\partial \lambda_1} \circ \frac{\partial}{\partial \lambda_1} = -\frac{\lambda_1}{3} \frac{\partial}{\partial \lambda_0}$

(5) Instead of Poincaré pairing on $T_t H$, in singularity theory we have the residue pairing on $T_\lambda \Lambda \cong Q_\lambda$:

$$\langle \phi, \psi \rangle_\lambda = \frac{1}{(2\pi i)^m} \oint \frac{\phi(z)\psi(z) dz_1 \cdots dz_m}{\frac{\partial f_\lambda}{\partial z_1} \cdots \frac{\partial f_\lambda}{\partial z_m}} = \sum_{z \in Cr(f_\lambda)} \frac{\phi(z)\psi(z)}{\det \left(\frac{\partial^2 f_\lambda}{\partial z_i \partial z_j}(z) \right)}$$

Now, let's turn around and look for GW analogues of some well-known concepts in singularity theory:

(6) In singularity theory, a very important role is played by the Hessian: for those values of λ for which f_λ has only nondegenerate critical points we can define $\Delta : Cr(f_\lambda) \to \mathbb{C}$ to be

$$\Delta(z) = \det \left(\frac{\partial^2 f_\lambda}{\partial z_i \partial z_j}(z) \right)$$

On the GW side, let $t \in H$ be such that the Frobenius algebra $(T_t H, \circ_t)$ is semisimple. Then we can define $\Delta : Spec_m(T_t H) \to \mathbb{C}$ by the formula

$$\langle \phi, \psi \rangle_t = \sum_{p \in Spec_m(T_t H)} \frac{\phi(p)\psi(p)}{\Delta(p)}$$

(compare with the formula in 5)

(7) A miniversal deformation $f(z, \lambda)$ generates an immersed Lagrangian submanifold $L \subset T^*\Lambda$ as follows. Let $\Sigma = \{(z, \lambda) | \frac{\partial f}{\partial z} = 0\}$ be the fibre-critical set of f ($\mathbb{C}^m \times \Lambda$ fibers over Λ). Then the mapping $\Sigma \to T^*\Lambda$ given by $(z, \lambda) \mapsto (\lambda, \frac{\partial f}{\partial \lambda})$ gives our Lagrangian immersion.

Example. For A_2, $\frac{\partial f(x,\lambda)}{\partial x} = 3x^2 + \lambda_1$, $\frac{\partial f}{\partial \lambda_1} = x$, $\frac{\partial f}{\partial \lambda_0} = 1$, hence $L \subset T^*\Lambda$ is given by equations

$$\begin{aligned} 3p^2 + 1 &= 0 \\ p_2 &= 1 \end{aligned}$$

In GW theory, the corresponding Lagrangian submanifold is defined to be $T^*H \supset L = \bigcup_t Spec_m(T_t H)$. Another description of L is given as follows: let $A = \sum_\alpha A_\alpha(t) dt_\alpha$ be the connection 1-form as in the definition of Frobenius structure. We have $dA = A \wedge A = 0$. Then

$$\det(A - \sum_\alpha p_\alpha dt_\alpha) = 0$$

gives a system of equations defining L. The above is a polynomial in $dt_1, \ldots, dt_N$, $N = \dim H^*(X)$; the condition $A \wedge A = 0$ guarantees that L exists, while $dA = 0$ implies that it is Lagrangian (at least at semisimple points). However, L may be singular.

Yet another description of L can be given in terms of the large quantum cohomology ring:

$$\mathbb{C}[L] = \frac{\mathbb{C}\left[\begin{array}{ccc} t_1, & \ldots & , t_N \\ p_1, & \ldots & , p_N \end{array}\right]}{\text{(rel's betw. } p\text{'s and } t\text{'s)}}$$