1. Quantum cohomology of flag manifolds (cont’d)

1.1. Notation. Let us finish up calculating relations in the quantum cohomology ring of a flag manifold \( X = G/B \), where \( G \) is a semisimple Lie group, \( B \) a Borel subgroup.

Let \( g' \) be the Langlands dual to \( g = \text{Lie}(G) \). We fix a Cartan decomposition of \( g' \):

\[
g' = n_- \oplus h \oplus n_+ \n\]

basis (e-vectors of \( \text{ad}_h \)) \( \{Y_\alpha\} \quad \{p_i\} \quad \{X_\alpha\} \)

roots (e-values) \(-\alpha \quad 0 \quad \alpha\)

Here \( \alpha \in h^* \) runs over all positive roots. We will denote by \( \{\alpha_i\} \) the set of simple roots; they form a basis of \( h^* \) dual to \( \{p_i\} \). We set \( b_- = n_- \oplus h, b_+ = h \oplus n_+ \).

Let \( Z \) be the center of \( U_{g'} \), the universal enveloping algebra, \( W \) the Weyl group.

1.2. Representation of \( Z \). Recall that we were about to construct a representation of \( U_{g'} \) in differential operators on the maximal torus \( T \) of \( G \) which, when restricted to \( Z \), will yield, after some modification, a set of commuting differential operators satisfying Kim’s lemma. Their symbols will then give us the relations in the quantum cohomology of \( X \). These operators are the integrals of the quantum Toda system.

The representation is constructed as follows. First, we pick a 1-dimensional representation \( L_+ \) of \( U_{n_+} \):

\[
X_\alpha \mapsto \begin{cases} 
  c_i & \text{if } \alpha = \alpha_i \text{ simple} \\
  0 & \text{otherwise}
\end{cases}
\]

Next, we factorize \( U_{g'} = U_b U_{n_+} \), i.e. we represent each element of \( U_{g'} \) as a sum of monomials of the form \( Y^kX^m \) (it is always possible to order the monomials in this way by using the relation \( xy - yx = [x, y] \)). Then we project thus factorised \( U_{g'} \) onto \( U_{b_-} \) along the kernel of \( L_+ \), and finally, we map \( U_{b_-} \) to the differential operators on the torus by:

\[
Y_\alpha \mapsto \begin{cases} 
  q_i & \text{if } \alpha = \alpha_i \text{ simple} \\
  0 & \text{otherwise}
\end{cases}
\]

\[
p_i \mapsto q_i \frac{\partial}{\partial q_i}
\]

Notice that this map is consistent with the commutation relations in \( U_{b_-} \) and \( \text{Diff}(T) \), and hence gives a well-defined algebra homomorphism. Therefore, the
composition
\[ Z \subset U_{q^*} = U_{b^-} U_{n^+} \xrightarrow{\ker L^+} U_{b^-} \xrightarrow{} Diff(T) \]
yields the sought-after representation of \( Z \). It sends the generators \( \Delta_1, \ldots, \Delta_r \) to commuting differential operators \( D_1, \ldots, D_r \).

1.3. Properties of \( D_i \)'s.

1.3.1. Polynomiality. It is clear from our construction that the resulting operators will be polynomial in \( q \) (no negative powers of \( q \) will occur).

1.3.2. \( W \)-invariance. Unfortunately, our operators will not, in general, have a \( W \)-invariant constant coefficient part. For instance, the Casimir element
\[ \Delta = \sum_{i,j} Q_{ij} p_i p_j + \sum_{\alpha > 0} y_{\alpha} x_{\alpha} + x_{\alpha} y_{\alpha} \in Z \]
will be mapped to an operator of the form
\[ D = \sum_{i,j} Q_{ij}(q_i \frac{\partial}{\partial q_j})(q_j \frac{\partial}{\partial q_j}) + \sum_k \lambda_k q_k + \sum_k \mu_k q_k \frac{\partial}{\partial q_k} \]
because we have to rewrite \( Y_\alpha X_\alpha + X_\alpha Y_\alpha = 2Y_\alpha X_\alpha + [X_\alpha, Y_\alpha] \), where \([X_\alpha, Y_\alpha] \in \mathfrak{h}\).

So let us investigate the \( W \)-invariance properties of our operators \( D_i \). By the Harish-Chandra isomorphism theorem, each \( \Delta_i = \sum \tilde{k}_i \tilde{p}^i \tilde{X}^m \) is uniquely determined by the part \( \tilde{k} = \tilde{m} = 0 \) (the Cartan part). In particular, commutativity of \( \Delta \) with \( \hat{h} \) implies that \( \sum (k_{\alpha_i}(-\alpha_i) + m_{\alpha_i} \alpha_i) = 0 \) for each monomial involving only simple roots (the others don’t matter as they are annihilated in our representation), hence \( k_{\alpha_i} = m_{\alpha_i} \) as \( \{\alpha_i\} \) form a basis of \( \mathfrak{h}^* \). So we see that the operators in the image of \( Z \) will have constant coefficients if and only if we choose \( L_+ = 0 \), since the surviving monomials will have as many \( X \)'s as \( Y \)'s.

Verma modules. For every \( \lambda \in \mathfrak{h}^* \) there exists a \( U_{q^*} \)-module \( V_\lambda \), called Verma module, characterized by the following property: \( \exists v \in V_\lambda \) such that
- \( n_+ v = 0 \)
- \( hv = (\lambda + \rho) h v \), \( \forall h \in \mathfrak{h} \), where \( \rho = \frac{1}{2} \sum_{\alpha < 0} \alpha \) is the Weyl vector
- \( V_\lambda \) is freely generated by the action of \( U_{n^-} \) on \( v \)

It is easy to see that \( V_\lambda \) is what is called infinitesimally irreducible, i.e. each \( \Delta \in Z \) acts on it by a scalar. It follows because, by definition of \( V_\lambda \) and the remark ending the preceding paragraph, the action of \( \Delta \) is determined by its Cartan part \( \Delta_{\text{Cartan}}(p_i) \), a polynomial in \( p_i \). Hence, \( v \) is an eigenvector of \( \Delta \) with eigenvalue \( \Delta_{\text{Cartan}}((\lambda + \rho)(p_i)) \). But since \( \Delta \in Z \), it acts on all of \( V_\lambda \) in this way, by definition of \( V_\lambda \).

The nontrivial fact about Verma modules we shall make use of is the following: if \( \lambda \) is a dominant weight (i.e. lies on a lattice point in a Weyl chamber), then \( V_{w\lambda} \) is a submodule of \( V_\lambda \) \( \forall w \in W \) (for generic \( \lambda \), \( V_\lambda \) is irreducible). It follows that, for \( \lambda \) dominant, \( \Delta \) acts by the same scalar on all \( V_{w\lambda} \), i.e.
\[ \Delta_{\text{Cartan}}(w\lambda + \rho) = \Delta_{\text{Cartan}}((\lambda + \rho)(p_i)) \quad \forall w \in W \]
Now, as we have established, the constant-coefficient part of the operator \( D \) corresponding to \( \Delta \in Z \) is \( \Delta_{\text{Cartan}}(-q_i \frac{\partial}{\partial q_i}) \). Therefore,
\[ D(q_i \frac{\partial}{\partial q_i}, 0) \exp(\mu(\ln q)) = \Delta_{\text{Cartan}}(\mu) \exp(\mu(\ln q)) \quad \forall \mu \in \mathfrak{h}^* \]
where $\mu(\ln q) = \sum \mu(p_i) \ln q_i$. From this we conclude that the operator $\exp(-\rho(\ln q))D \exp(\rho(\ln q))$ will have a $W$-invariant constant-coefficient part. For instance, the Casimir element will go to the Hamiltonian of the Toda lattice under this correspondence.

References:
B. Kostant (Invent. 1928)
Semenov-Tian-Shansky (Encyclopedia of Math. Sciences)

1.4. Open questions related to $QH^*(G/B)$. We have just calculated a presentation of $QH^*(G/B)$ in terms of generators and relations; as we learned in an earlier lecture, it is not enough. Here is a list of things we don’t know about the quantum cohomology of flag manifolds:

1.4.1. All structure constants in a linear basis, except in the following special cases:

$SL$: Known due to (S. Gelfand, S. Fomin and A. Postnikov).

$SO^\text{odd}$: Conjecture by Maeno.

In particular, we can’t apply Konstevich-Manin to calculate the large $QH^*$.

1.4.2. Even for $SL$ - what is the meaning of the representation-theoretic formulas (Toda lattices) and what is the ”actual” relation between $QH^*(G/B)$ and the representation theory of $g'$?

1.4.3. Partial flag manifolds. These are homogeneous spaces of the type $G/P$ where $P$ is a parabolic subgroup.

Example. $G = SL_N$. A partial flag is a flag of the form

\[
\{0\} \subset \mathbb{C}^{n_0} \subset \mathbb{C}^{n_0+n_1} \subset \cdots \subset \mathbb{C}^{n_0+n_1+\cdots+n_k} = \mathbb{C}^N,
\]

where $n_i$ are fixed positive integers adding up to $N$. The manifold of such flags is a homogeneous space $X = G/P$ where $P$ is the subgroup of block-upper-triangular matrices with blocks of size $n_i$. Complete flags correspond to all $n_i = 1$.

The (ordinary) cohomology ring of $X$ is computed in a similar fashion to that of a complete flag manifold. We have $k+1$ tautological $\mathbb{C}^{n_i}$-bundles over $X$ whose Chern polynomials

\[
p_{n_i}(x) = x^{n_i} + c_1^{(i)} x^{n_i-1} + \cdots + c_{n_i}^{(i)},
\]

satisfy the relation

\[
P_N(x) = \prod_i p_{n_i} = x^N
\]

by virtue of the sum of the tautological bundles being trivial of rank $N$. Then $H^*(X)$ is multiplicatively generated by the Chern classes $c_j^{(i)}$ subject to the above relation.

In the quantum deformation of $H^*(X)$ there are $k$ additional generators $q_1, \ldots, q_k$, and the relation is given by $P_N(x, q) = x^N$, where $P_N$ is determined by

\[
p_{n_0} + \frac{q_1}{p_{n_1}} + \frac{q_2}{p_{n_2}} + \cdots + \frac{q_k}{p_{n_k}} = P_N(x, q)
\]

\[
Q_{N-n_0}(x, q)
\]
In particular, the case \( k = 1 \) corresponds to Grassmanians, for which the relation is

\[
\frac{p_{n_0}p_{n_1} + q_1}{p_{n_1}} = \frac{x^{n_0 + n_1}}{p_{n_1}},
\]

as we computed before.

**References:**
- Astashkevich-Sadov, B. Kim (heuristic)
- Ciocan-Fontaine, B. Kim (proof)
- D. Peterson (\( QH^*(G/P) \) in geometrical terms of Toda lattices)

1.4.4. Flag manifolds associated to loop groups. If \( G \) is a Lie group, its loop group is \( LG = \text{Maps}(S^1 \to G) \), where by "maps" one usually means smooth, analytic or polynomial ones. In what follows we shall restrict ourselves to the special case of \( G = SL_n \).

Let us consider the vector space \( H = H^- \oplus H^+ \) of Laurent polynomials in one variable \( z \), where \( H^- = \langle \ldots, z^{-2}, z^{-1} \rangle \), \( H^+ = \langle 1, z, z^2, \ldots \rangle \). The semi-infinite Grassmanian \( G_{2\infty}^+ \) consists of subspaces of \( H \) spanned by Laurent polynomials whose degree is bounded below:

\[
G_{2\infty}^+ = \{ W \subset H | z^k H^+ \subset W \subset z^{-k} H^+ \text{ for some } k \}
\]

We shall be interested in flags of the form

\[
\cdots \supset W_{-2} \supset W_{-1} \supset W_0 \supset W_1 \supset W_2 \supset \cdots
\]

where all \( W_i \in G_{2\infty}^+ \). We shall be interested in flags of this type form a flag manifold of \( LSL_n \).

**Observation** (Atiyah): Fixed degree compact holomorphic curves passing through a given point in this flag manifold come in finite parametric families (in contrast, in \( \mathbb{C}P^\infty \) there is an infinite-dimensional family of straight lines). As in studying holomorphic spheres in ordinary flag manifolds, we look at the fibrations

\[
\begin{align*}
W_0 \supset \cdots \supset W_1 &\supset \cdots \supset W_n = z^n W_0 \\
\mathbb{C}P^1 \\
W_0 \supset \cdots \supset W_{i-1} &\supset W_{i+1} \cdots \supset W_n = z^n W_0
\end{align*}
\]

As in the ordinary case, the only curves of degree equal to that of a fibre are the fibres themselves, hence there is an infinite-dimensional space of such curves. However, if we require the curve to pass through a given point, only one fibre will remain (degrees are different for different \( i \)'s). Thus one is motivated to define quantum multiplication as

\[
< \phi_\alpha \circ \phi_\beta, \phi_\gamma > = \sum q^T n_d
\]

where \( \phi_\alpha \) and \( \phi_\beta \) are cycles of finite codimension, \( \phi_\gamma \) is a cycle of finite dimension, and \( n_d = \#((\mathbb{C}P^1, 0, 1, \infty) \to (X, \phi_\alpha, \phi_\beta, \phi_\gamma)) \). By Atiyah’s observation, this makes sense, and so we can define quantum multiplication even in the absence of Poincaré pairing. The resulting computation will be related to another integrable system, the periodic Toda lattice.

**Remark** (A. Kogan). This flag manifold is a limit of finite-dimensional ones, so perhaps one can count curves lying completely within finite-dimensional pieces.