TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 13

A. GIVENTAL

1. Quantum cohomology of flag manifolds (cont'd)

1.1. **Notation.** Let us finish up calculating relations in the quantum cohomology ring of a flag manifold $X = G/B$, where G is a semisimple Lie group, B a Borel subgroup.

Let $\mathfrak{g}'$ be the Langlands dual to $\mathfrak{g} = \mathrm{Lie}(G)$. We fix a Cartan decomposition of $\mathfrak{g}'$:

$$\begin{array}{rccccc} \mathfrak{g}' = & \mathfrak{n}_- & \oplus & \mathfrak{h} & \oplus & \mathfrak{n}_+ \\ \text{basis (e-vectors of } ad_{\mathfrak{h}}) & \{Y_\alpha\} & & \{p_i\} & & \{X_\alpha\} \\ \text{roots (e-values)} & -\alpha & & 0 & & \alpha \end{array}$$

Here $\alpha \in \mathfrak{h}^*$ runs over all positive roots. We will denote by $\{\alpha_i\}$ the set of simple roots; they form a basis of $\mathfrak{h}^*$ dual to $\{p_i\}$. We set $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$, $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$.

Let Z be the center of $U_{\mathfrak{g}'}$, the universal enveloping algebra, W the Weyl group.

1.2. **Representation of Z.** Recall that we were about to construct a representation of $U_{\mathfrak{g}'}$ in differential operators on the maximal torus T of G which, when restricted to Z, will yield, after some modification, a set of commuting differential operators satisfying Kim's lemma. Their symbols will then give us the relations in the quantum cohomology of X. These operators are the integrals of the quantum Toda system.

The representation is constructed as follows. First, we pick a 1-dimensional representation L_+ of $U_{\mathfrak{n}_+}$:

$$X_\alpha \mapsto \begin{cases} c_i & \text{if } \alpha = \alpha_i \text{ simple} \\ 0 & \text{otherwise} \end{cases}$$

Next, we factorize $U_{\mathfrak{g}'} = U_{\mathfrak{b}_-} U_{\mathfrak{n}_+}$, i.e. we represent each element of $U_{\mathfrak{g}'}$ as a sum of monomials of the form $\vec{Y}^{\vec{k}} \vec{p}^{\vec{l}} \vec{X}^{\vec{m}}$ (it is always possible to order the monomials in this way by using the relation $xy - yx = [x, y]$). Then we project thus factorised $U_{\mathfrak{g}'}$ onto $U_{\mathfrak{b}_-}$ along the kernel of L_+, and finally, we map $U_{\mathfrak{b}_-}$ to the differential operators on the torus by:

$$\begin{array}{rcl} Y_\alpha & \mapsto & \begin{cases} q_i & \text{if } \alpha = \alpha_i \text{ simple} \\ 0 & \text{otherwise} \end{cases} \\ p_i & \mapsto & q_i \frac{\partial}{\partial q_i} \end{array}$$

Notice that this map is consistent with the commutation relations in $U_{\mathfrak{b}_-}$ and $Diff(T)$, and hence gives a well-defined algebra homomorphism. Therefore, the

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composition

$$Z \subset U_{\mathfrak{g}'} = U_{\mathfrak{b}_-} U_{\mathfrak{n}_+} \overset{\ker L_+}{\longrightarrow} U_{\mathfrak{b}_-} \longrightarrow Diff(T)$$

yields the sought-after representation of Z. It sends the generators $\Delta_1, \dots \Delta_r$ to commuting differential operators $D_1, \dots D_r$.

1.3. **Properties of D_i's.**

1.3.1. *Polynomiality.* It is clear from our construction that the resulting operators will be polynomial in q (no negative powers of q will occur).

1.3.2. *W-invariance.* Unfortunately, our operators will not, in general, have a W-invariant constant coefficient part. For instance, the Casimir element

$$\Delta = \sum_{i,j} Q_{ij} p_i p_j + \sum_{\alpha > 0} \frac{y_\alpha x_\alpha + x_\alpha y_\alpha}{< x_\alpha, y_\alpha >} \in Z$$

will be mapped to an operator of the form

$$D = \sum_{i,j} Q_{ij} (q_i \frac{\partial}{\partial q_i})(q_j \frac{\partial}{\partial q_j}) + \sum_k \lambda_k q_k + \sum_k \mu_k q_k \frac{\partial}{\partial q_k}$$

because we have to rewrite $Y_\alpha X_\alpha + X_\alpha Y_\alpha = 2 Y_\alpha X_\alpha + [X_\alpha, Y_\alpha]$, where $[X_\alpha, Y_\alpha] \in \mathfrak{h}$

So let us investigate the W-invariance properties of our operators D_i. By the Harish-Chandra isomorphism theorem, each $\Delta_i = \sum \vec{Y}^{\vec{k}} \vec{p}^{\vec{l}} \vec{X}^{\vec{m}}$ is uniquely determined by the part $\vec{k} = \vec{m} = 0$ (the Cartan part). In particular, commutativity of Δ with $\mathfrak{h}$ implies that $\sum (k_{\alpha_i}(-\alpha_i) + m_{\alpha_i} \alpha_i) = 0$ for each monomial involving only simple roots (the others don't matter as they are annihilated in our representation), hence $k_{\alpha_i} = m_{\alpha_i}$ as $\{\alpha_i\}$ form a basis of $\mathfrak{h}^*$. So we see that the operators in the image of Z will have constant coefficients if and only if we choose $L_+ = 0$, since the surviving monomials will have as many X's as Y's.
Verma modules. For every $\lambda \in \mathfrak{h}^*$ there exists a $U_{\mathfrak{g}'}$-module V_λ, called *Verma module*, characterized by the following property: $\exists v \in V_\lambda$ such that

- $\mathfrak{n}_+ v = 0$
- $hv = (\lambda + \rho)(h)v$, $\forall h \in \mathfrak{h}$, where $\rho = \frac{1}{2} \sum_{\alpha < 0} \alpha$ is the Weyl vector
- V_λ is freely generated by the action of $U_{\mathfrak{n}_-}$ on v

It is easy to see that V_λ is what is called *infinitesimally irreducible*, i.e each $\Delta \in Z$ acts on it by a scalar. It follows because, by definition of V_λ and the remark ending the preceding paragraph, the action of Δ is determined by its Cartan part $\Delta_{Cartan}(p_i)$, a polynomial in p_i. Hence, v is an eigenvector of Δ with eigenvalue $\Delta_{Cartan}((\lambda + \rho)(p_i))$. But since $\Delta \in Z$, it acts on all of V_λ in this way, by definition of V_λ.

The nontrivial fact about Verma modules we shall make use of is the following: if λ is a *dominant* weight (i.e. lies on a lattice point in a Weyl chamber), then $V_{w\lambda}$ is a submodule of $V_\lambda \quad \forall w \in W$ (for generic λ, V_λ is irreducible). It follows that, for λ dominant, Δ acts by the same scalar on all $V_{w\lambda}$, i.e.

$$\Delta_{Cartan}(w\lambda + \rho) = \Delta_{Cartan}(\lambda + \rho) \qquad \forall w \in W$$

Now, as we have established, the constant-coefficient part of the operator D corresponding to $\Delta \in Z$ is $\Delta_{Cartan}(-q_i \frac{\partial}{\partial q_i})$. Therefore,

$$D(q\frac{\partial}{\partial q}, 0) \exp(\mu(\ln q)) = \Delta_{Cartan}(\mu) \exp(\mu(\ln q)) \quad \forall \mu \in \mathfrak{h}^*,$$

where $\mu(\ln q) = \sum \mu(p_i) \ln q_i$. From this we conclude that the operator $\exp(-\rho(\ln q)) D \exp(\rho(\ln q))$ will have a W-invariant constant-coefficient part. For instance, the Casimir element will go to the Hamiltonian of the Toda lattice under this correspondence.

References:

B. Kostant (Invent. 1928)

Semenov-Tian-Shansky (Encyclopedia of Math. Sciences)

1.4. **Open questions related to $QH^*(G/B)$.** We have just calculated a presentation of $QH^*(G/B)$ in terms of generators and relations; as we learned in an earlier lecture, it is not enough. Here is a list of things we don't know about the quantum cohomology of flag manifolds:

1.4.1. All structure constants in a linear basis, except in the following special cases:

SL**:** Known due to (S. Gelfand, S. Fomin and A. Postnikov).

$SO_{\mathbf{odd}}$**:** Conjecture by Maeno.

In particular, we can't apply Kontsevich-Manin to calculate the large QH^*.

1.4.2. Even for SL - what is the meaning of the representation-theoretic formulas (Toda lattices) and what is the "actual" relation between $QH^*(G/B)$ and the representation theory of $\mathfrak{g}'$?

1.4.3. *Partial flag manifolds.* These are homogeneous spaces of the type G/P where P is a parabolic subgroup.

Example. $G = SL_N$. A partial flag is a flag of the form

$$\{0\} \subset \mathbb{C}^{n_0} \subset \mathbb{C}^{n_0+n_1} \subset \cdots \subset \mathbb{C}^{n_0+n_1+\cdots n_k} = \mathbb{C}^N,$$

where n_i are fixed positive integers adding up to N. The manifold of such flags is a homogeneous space $X = G/P$ where P is the subgroup of block-upper-triangular matrices with blocks of size n_i. Complete flags correspond to all $n_i = 1$.

The (ordinary) cohomology ring of X is computed in a similar fashion to that of a complete flag manifold. We have $k+1$ tautological $\mathbb{C}^{n_i}$-bundles over X whose Chern polynomials

$$p_{n_i}(x) = x^{n_i} + c_1^{(i)} x^{n_i - 1} + \cdots + c_{n_i}^{(i)}$$

satisfy the relation

$$P_N(x) = \prod_i p_{n_i} = x^N$$

by virtue of the sum of the tautological bundles being trivial of rank N. Then $H^*(X)$ is multiplicatively generated by the Chern classes $c_j^{(i)}$ subject to the above relation.

In the quantum deformation of $H^*(X)$ there are k additional generators $q_1, \ldots, q_k$, and the relation is given by $P_N(x,q) = x^N$, where P_N is determined by

$$p_{n_0} + \cfrac{q_1}{p_{n_1} + \cfrac{q_2}{\ddots + \cfrac{q_k}{p_{n_k}}}} = \frac{P_N(x,q)}{Q_{N-n_0}(x,q)}$$

In particular, the case $k = 1$ corresponds to Grassmanians, for which the relation is

$$\frac{p_{n_0} p_{n_1} + q_1}{p_{n_1}} = \frac{x^{n_0+n_1}}{p_{n_1}},$$

as we computed before.

References:

Astashkevich-Sadov, B. Kim (heuristic)
Ciocan-Fontaine, B. Kim (proof)
D. Peterson ($QH^*(G/P)$ in geometrical terms of Toda lattices)

1.4.4. *Flag manifolds associated to loop groups.* If G is a Lie group, its *loop group* is $\mathcal{L}G = Maps(S^1 \to G)$, where by "maps" one usually means smooth, analytic or polynomial ones. In what follows we shall restrict ourselves to the special case of $G = SL_n$.

Let us consider the vector space $H = H_- \oplus H_+$ of Laurent polynomials in one variable z, where $H_- =< \dots, z^{-2}, z^{-1} >$, $H_+ =< 1, z, z^2, \dots >$. The *semi-infinite Grassmanian* $G_{\frac{1}{2}\infty}$ consists of subspaces of H spanned by Laurent polynomials whose degree is bounded below:

$$G_{\frac{1}{2}\infty} = \{W \subset H | z^k H_+ \subset W \subset z^{-k} H_+ \quad \text{for some } k\}$$

We shall be interested in flags of the form

$$\cdots \supset W_{-2} \supset W_{-1} \supset W_0 \supset W_1 \supset W_2 \supset \cdots$$

where all $W_i \in G_{\frac{1}{2}\infty}$, $\dim W_i/W_{i+1} = 1$ and $W_{i+n} = z^n W_i$. Flags of this type form a flag manifold of $\mathcal{L}SL_n$.

Observation (Atiyah): Fixed degree compact holomorphic curves passing through a given point in this flag manifold come in finite parametric families (in contrast, in $\mathbb{C}\mathbf{P}^\infty$ there is an infinite-dimensional family of straight lines). As in studying holomorphic spheres in ordinary flag manifolds, we look at the fibrations

$$\begin{array}{ccc} W_0 \supset \cdots \supset & W_i & \supset \cdots \supset W_n = z^n W_0 \\ & \Big\downarrow & \mathbb{C}\mathbf{P}^1 \\ W_0 \supset \cdots W_{i-1} & \supset & W_{i+1} \cdots \supset W_n = z^n W_0 \end{array}$$

As in the ordinary case, the only curves of degree equal to that of a fibre are the fibres themselves, hence there is an infinite-dimensional space of such curves. However, if we require the curve to pass through a given point, only one fibre will remain (degrees are different for different i's). Thus one is motivated to define quantum multiplication as

$$< \phi_\alpha \circ \phi_\beta, \phi_\gamma^\vee >= \sum \vec{q}^{\vec{d}} n_d$$

where ϕ_α and ϕ_β are cycles of finite codimension, $\phi_\gamma^\vee$ is a cycle of finite dimension, and $n_d = \#((\mathbb{C}\mathbf{P}^1, 0, 1, \infty) \to (X, \phi_\alpha, \phi_\beta, \phi_\gamma^\vee))$. By Atiyah's observation, this makes sense, and so we can define quantum multiplication even in the absence of Poincaré pairing. The resulting computation will be related to another integrable system, the *periodic Toda lattice*.

Remark (A. Kogan). This flag manifold is a limit of finite-dimensional ones, so perhaps one can count curves lying completely within finite-dimensional pieces.