TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 12

A. GIVENTAL

Let the notation be as in the previous lecture. Let G be any complex semisimple Lie group unless otherwise specified.

Let us recall that we are looking for the differential operators $D_1, \cdots, D_r$ on the maximal torus which commute with the operator $Q(q\frac{\partial}{\partial q}) - \Sigma Q_{kk}q_k$. Our approach will be to construct integrable system, called Toda lattice, with the Hamiltonian $Q(q\frac{\partial}{\partial q}) - \Sigma Q_{kk}q_k$. Then the operators D_i's will be the integrals of motion of the system.

The phase space of the Toda lattice is the cotangent bundle of the Cartan subalgebra h of $\mathfrak{g}$. The phase space is then equal to $h \oplus h^*$ which is isomorphis to T^*h. The symplectic form is the standard symplectic form of a cotangent bundle. The Hamiltonian is defined to be

$$H(x,t) = (x,x) - \sum_{\text{simple roots } \alpha_i} (\alpha_i, \alpha_i) exp(\alpha_i(x)).$$

There is a symplectomorphism (a morphism preserving the symplectic form) from T^*h to some orbit of the coadjoint action of B_- on the dual of its Lie algebra identified with $U\mathfrak{b}_+$. Let us describe this map. Let x_{α_i}'s be the canonical basis elements of $\mathfrak{b}_+$ corresponding to simple roots (in the case of SL_{r+1} they correspond to matrices with 1 above diagonal at one place and zeros elsewhere). Let $e = \sum x_{\alpha_i}$. It is easy to see from the definition of the coadjoint action that the orbit of e is $O = h \oplus \sum \mathbb{C}^* x_{\alpha_i} = T^*T$, where T is a maximal torus. The equality above is a symplectic isomorphism from the orbit and the cotangent bundle to the torus, where the orbit has Kirillov-Kostant bracket, and the bundle has the usual bracket.

The map from $h \oplus h^*$ to $T^*T = h \oplus \sum \mathbb{C}^{*r}$ is given by identity on the first factor and the exponentiating on the second factor.

Claim : This map is a symplectomorphism.

To solve the problem in the new setting we have to find r independent functions on $\mathfrak{b}_+$ which Poisson commute between themselves. The pull-back one of this functions to $h \oplus h^*$ should be our original Hamiltonian H. Such (Casimir) functions always exist - they are the functions which are invariant under the adjoint action of B_-. This is the origin of integrability of Toda lattice.

Example. Let $G = SL_{r+1}$. Let $t_1, \cdots, t_r$ be the coordinate functions on the dual of Cartan subalgebra, $t_0 = -\sum t_i$. Let $x_0 = \frac{\partial}{\partial t_0}, \cdots, x_r = \frac{\partial}{\partial t_r}$ be functions on the Cartan subalgebra h^*, $\sum x_i = 0$. The symplectic form is $\sum_{i=1}^r dx_i \wedge dt_i$. The cotangent bundle $T^*T = h \oplus \mathbb{C}^{*r}$, which is the orbit of the coadjoint action), has coordinates $x_i, q_i, i = 1, \cdots, r$, where $q_k = e^{t_{k-1}-t_k}$. The Hamiltonian of the Toda

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lattice is then

$$H = \frac{1}{2}\sum_{i=0}^{r} \frac{\partial^2}{\partial {t_i}^2} - \sum_{i=1}^{r} e^{t_{k-1}-t_k} = \frac{1}{2}\sum x_i^2 - \sum e^{t_{k-1}-t_k}.$$

This Hamiltonian coincides with our operator $Q(q\frac{\partial}{\partial q}) - \Sigma Q_{kk} q_k$ defined in the previous lecture.

Remark. Our Toda lattice describes the dynamics of the system of $r+1$ particles on the line with the potential $\sum e^{t_{k-1}-t_k}$.

Integral of motions $D_1, \cdots, D_r$ are obtained from the following formula:

$$det \begin{pmatrix} x + \hbar\frac{\partial}{\partial t_0} & e^{t_0-t_1} & 0 & 0 \\ -1 & * & * & 0 \\ 0 & * & * & e^{t_{r-1}-t_r} \\ 0 & 0 & -1 & x+\hbar\frac{\partial}{\partial t_r} \end{pmatrix} = x^{r+1} + D_1 x^r + \cdots D_{r+1}.$$

It is easy to check that D_i's indeed commute with H, which is equal to the half of the trace of the square of the above matrix. Let $\tilde{\sigma}_i = D_i(q=0)$.

Conclusion.

$$QH^*(X) = \mathbb{Q}[x_0, \cdots, x_r, q_1, \cdots, q_r]/(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_{r+1})$$

Remark. The subvariety of the configuration space defined by the system of equations $\{\tilde{\sigma}_i = 0\}$ is an invariant Lagrangian variety of the Toda lattice. Quantum cohomology is just the algebra of functions on this variety.

By PBW theorem the space of polynomials on $\mathfrak{b}_+$ is isomorphic as vector spaces to $U(\mathfrak{b}_+)$. Let us understand the integrability of Toda lattice in terms of the universal enveloping algebra.

Let G' be the group Langlands dual to G, N_-, N_+ - nilpotent subgroups (lower and upper triangular matrices). The maximal torus $T \simeq N_- \ G'/N_+$. Let $\chi_\epsilon : N_\epsilon \to \mathbb{C}^*$, be 1-dimensional generic representations of $N_\epsilon, \epsilon = +$ or $-$.

Recall that Langlands duality of Lie algebras exchanges B and C series and doesn't change other series. Let $\mathfrak{g}'$ be the Lie algebra Langlands dual to the Lie algebra of G. The universal enveloping algebra $U(\mathfrak{g}')$ is naturally represented in the space of left invariant differential operators on G'. The center $Z\mathfrak{g}'$ has the representation by differential biinvariant operators.

Suppose we want to restrict the action of these operators on the maximal torus only. To do this we restrict them on the space of functions on G' which can be continued from the maximal torus using characters χ_+ and χ_-.

Namely, let L_χ be the sheaf of functions on G' satisfying the following property:

$$f(n_-^{-1} g n_+) = \chi_-^{-1}(n_-) f(g) \chi_+(n_+).$$

$U\mathfrak{g}'$ (resp. $Z\mathfrak{g}'$) act by left invariant (resp. biinvariant) differential operators on L_χ and, therefore, on the functions on the maximal torus T. Our operator H is in the image of this representation. Therefore to construct the operators commuting with H is enough to describe the image of this representation of $Z\mathfrak{g}'$.

By Harish-Chandra theorem $Z\mathfrak{g}'$ is isomorphic to the W-invariant polynomials $\mathbb{C}[h^*]^W$ on h. The algebra $\mathbb{C}[h^*]^W$ is a polynomial algebra on $r = rankh$ generators. Their preimages in $Z\mathfrak{g}'$ are called Casimir elements. They give us exactly commuting differential operators on the maximal torus.

Let us look at the explicite algebraic picture. Let p_i's be the generators of the Cartan subalgebra, x_α (resp. y_α), $\alpha > 0$, be the generators of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) (upper and lower triangular matrices). Also let $q_1, \cdots, q_r$ be coordinate functions on the maximal torus T as before.

We have the following sequence:

$$Z\mathfrak{g}' \to U(\mathfrak{g}') = U(\mathfrak{b}_-) \otimes U(\mathfrak{n}_+) \to U(\mathfrak{b}_-)$$

Let us explain the second arrow. The character χ_+ of N_+ is the exponentiation of some representation $L_+ : \mathfrak{n}_+ \to \mathbb{C}$. The second arrow of the above sequence means factoring by the relations $n_+ = L_+(n_+), n_+ \in \mathfrak{n}_+$. Let L_- have the similar meaning. Choosing L_- is equivalent to choosing the point in $\mathfrak{b}_+$ which belong to the orbit isomorphic to T^*T (see beginning of this lecture). We choose L_- as there.

This gives us the following representation of $U\mathfrak{g}'$ (and, therefore, $Z\mathfrak{g}'$) in the differential operators on T:

$$\begin{gathered} p_i \to -q_i \frac{\partial}{\partial q_i}, \\ x_\alpha, y_\alpha \to 0, \text{ if } \alpha \text{ is not a simple root }, \\ x_{\alpha_i} \to \text{ nonzero constant } c_i, \\ y_{\alpha_i} \to q_i \times, \end{gathered}$$

where α_i's are simple roots.

Let $C \in Z\mathfrak{g}'$ be the Casimir element

$$C = \sum Q_{ij} p_i p_j + \sum_{\alpha > 0} \frac{y_\alpha x_\alpha + x_\alpha y_\alpha}{< x_\alpha, y_\alpha >}$$

Its image in this representation is the Laplacian

$$\Delta = \sum Q_{ij} (q_i \frac{\partial}{\partial q_i})(q_j \frac{\partial}{\partial q_j}) + \sum c_k q_k + \sum \mu_k q_k \frac{\partial}{\partial q_k}.$$

We can observe that our Hamiltonian

$$H = e^{-\rho ln q} \Delta e^{\rho ln q}.$$