TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 11

A. GIVENTAL

1. Quantum cohomology of flag manifolds.

1.1. **Brief review of the cohomology theory of flag manifolds.** Let G be complex semisimple Lie group, B - its Borel subgroup. From now on we'll consider the example $G = SL_{r+1}(\mathbb{C})$, but the results can be easily generalized to arbitrary G.

In our case B can be identified with the group of upper-triangular matrices with determinant 1. The manifold $X = G/B$ can be identified with the space of flags:

$$X = \{\cdots \mathbb{C}^i \subset \mathbb{C}^{i+1} \cdots\}$$

(G acts naturally on this space, B is a stabilizer of some point.)

Let us calculate $H^*(X)$. Let $l_i, i = 0 \cdots r$, be the following tautological line bundles:

$$\begin{array}{c} \mathbb{C}^{i+1}/\mathbb{C}^i \\ \big\downarrow \\ X \end{array}$$

Let $x_0, \cdots, x_r$ be their negative first Chern classes. From the identity

$$(x - x_0) \cdots (x - x_r) = x^{r+1}$$

we deduce that

$$H^*(X) = \mathbb{Z}[x_0, \cdots, x_r]/(\sigma_1, \cdots, \sigma_{r+1}),$$

where σ_i is the i^{th} elementary symmetric function of $r+1$ variables.

In particular it follows that $H^2(X) \simeq \mathbb{Z}^r$ (because $\Sigma x_i = 0$).

Remark. Let us consider the case of general G now. According to the Borel-Bott-Weil theorem there is one-to-one correspondence between irreducible highest weight representations of the Lie algebra $\mathfrak{g}$ of G and line bundles on $X = G/B$. Let us recall this correspondence. Suppose we are given a representation of $\mathfrak{g}$ or, equivalently, a character χ of the maximal torus of G. There is a natural projection from B to the maximal torus $(\mathbb{C}^*)^r$, so we can pullback χ to B. Let $\mathbb{C}_\chi$ be one-dimensional representation of B with character χ. Then we can construct the following line bundle:

$$\begin{array}{c} G \times_B \mathbb{C}_\chi \\ \big\downarrow \\ X \end{array}$$

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Notes taken by Alexander Kogan.

Conversely, given a line bundle l over X, the set of its holomorphic sections has a natural structure of the highest weight representation of $\mathfrak{g}$.

We know that line bundles over X are uniquely identified by their Chern classes. Therefore we arrive at the conclusion: weight lattice of $\mathfrak{g} = H^2(X,\mathbb{Z})$. As a consequence we have the action of the Weyl group W on $H^2(X,\mathbb{Z})$.

Let $p_1, \cdots, p_r$ be the basis of second cohomology. Now we can describe the cohomology of X:

$$H^*(X) = \mathbb{Z}[p_1, \cdots, p_r]/(W - \text{invariant polynomials in } p_i^\prime s).$$

It is convenient if p_i's correspond to the fundamental weights. The corresponding line bundles are called fundamental line bundles. Returning to the case of $SL_{r+1}(\mathbb{C})$ we can observe that $p_i = x_0 + \cdots + x_{i-1} = c_1(\Lambda^i(\mathbb{C}^*)^i)$, i.e. p_i's are the first Chern classes of fundamental line bundles which correspond to the representations in the spaces of exterior forms (fundamental representations).

1.2. **Quantum cohomology of flag varieties.** The main reference is the paper of Bumsig Kim "Quantum Cohomology Of Flag Manifolds G/B And Quantum Toda Lattices.", alg-geom/9607001.

We need some information about the holomorphic curves on flag manifolds. We have the following bundle:

$$\begin{array}{c} X = \{\cdots \mathbb{C}^{i-1} \subset \mathbb{C}^i \subset \mathbb{C}^{i+1} \cdots\} \\ \big\downarrow \mathbb{C}\mathbf{P}^1 \\ \{\cdots \mathbb{C}^{i-1} \subset \mathbb{C}^{i+1} \cdots\} \end{array}$$

(i.e. we skip $\mathbb{C}^i$).

Let us denote the generators of homology classes by $\mathbf{1}_1, \cdots, \mathbf{1}_r$. They form the basis of $H^2(X)$ dual to $p_1, \cdots, p_r$. The holomorphic curves in X with degrees $\mathbf{1}_i$ are the fibers of the above bundle. This is because their degree (i.e. homological class $\mathbf{1}_i$) is 0 in the image. They cannot be multiple-covers of the fibers also by degree considerations. Therefore $X_{0,1,\mathbf{1}_i} = X$ and $X_{0,0,\mathbf{1}_i} = \{\cdots \mathbb{C}^{i-1} \subset \mathbb{C}^{i+1} \cdots\}$. The map above is just the forgetful map.

Let us compute the quantum cohomology ring

$$QH^*(X) = \mathbb{Q}[p_1, \cdots, p_r, q_1, \cdots, q_r]/(\text{relations}).$$

Claim: $deg(p_i) = 1, deg(q_i) = 2$.

Proof. $dimX = dimX_{0,1,\mathbf{1}} = deg(q_i) - 2 + dim(X) \Rightarrow deg(q_i) = 2$. QED.

Explanation. Let us understand the first Chern character of the tangent bundle of X. The tangent bundle is the sum of line bundles whose characters correspond to the positive roots. For example, in the case of $G = SL_{r+1}$ the tangent space to G is GL_{r+1}, and the tangent space to G/B is the lower-triangular part of GL_{r+1}, which is the sum of matrix elements.

Therefore $c_1(T_X) = \sum_{\alpha>0} = 2\rho$, where ρ is the Weyl vector. In the above statement I implicitely identified the weight lattice with the lattice $H^2(X,\mathbb{Z})$. From the definition of ρ we know that $(\rho, p_i) = 1$, because p_i's are identified with fundamental weights and $\rho = p_1 + \cdots p_r$.

Let us consider the general case now. Let $Q(p) = \sum Q_{ij} p_i p_j$ be the relation in $H^2(X)$. We want to compute $\Sigma Q_{ij} p_i \circ p_j$. The degree of this expression is 2, therefore it must a linear combination of q_i's or $p_i p_j$'s. But if $q_i = 0$ the expression must vanish in cohomology, so it is equal to $\sum c_k q_k$, where c_k's are some constants. Let's compute them. Evaluation of the above expression on a point gives us:

$$\sum < p_i \circ p_j, [pt] >= \sum c_k q_k.$$

By looking at the term of degree $\mathbf{1}_k$ we get:

$$c_k = \sum (p_i, p_j, [pt])_{0,3,\mathbf{1}_k}.$$

From the geometric picture we see that $(p_i, p_j, [pt])_{0,3,\mathbf{1}_k} = 1$ if $i = j$, and 0 otherwise. Therefore we get the answer: $c_k = Q_{kk}$. Thus we get the relation

$$\Sigma Q_{ij} p_i \circ p_j = \sum Q_{kk} q_k \tag{1}$$

Lemma 1.

$$\textit{If } D(\hbar q\frac{\partial}{\partial q}, q, \hbar)\vec{J}(q, lnq, \hbar^{-1}) = 0 \textit{ then } D(p\circ, q, 0) = 0 \textit{ in } QH^*(X),$$

where $D(\hbar q\frac{\partial}{\partial q}, q, \hbar)$ is some differential operator that is polynomial in $\hbar q\frac{\partial}{\partial q}$ and a power series in q and $\hbar$.

Thus the relations in cohomology are symbols of differential operators which annihilate $\vec{J}$. Question: what differential operator gives the relation 1. Answer:

Theorem 1. $[Q(\hbar q\frac{\partial}{\partial q}) - \Sigma Q_{kk} q_k]\vec{J} = 0$

Proof. Let S be the fundamental solution matrix, $\vec{J}$ - its first row. Let us denote $H = Q(\hbar q\frac{\partial}{\partial q}) - \Sigma Q_{kk} q_k$. Applying H to the whole S we get:

$$HS = \hbar \sum_{i,j} Q_{ij}(q_i \frac{\partial}{\partial q_i}(p_j\circ))S.$$

The first row in $(p_j\circ)S$ is constant because for any cohomology class ϕ_ν

$$< \mathbf{1}, p_j \circ \phi_\nu >=< p_j \circ \mathbf{1}, \phi_\nu >=< p_j, \phi_\nu >=$$
$$= \text{intersection index of two classes} \;\Rightarrow\; \text{constant}$$

Therefore differentiating the first line of $(p_j\circ)$S we get 0.

QED.

Corollary. $\vec{J} = e^{pln(q)} \Sigma_{d\in\Lambda} P_d(\hbar^{-1}) q^d$ is uniquely determined by conditions $H\vec{J} = 0$ and $P_0 = 0$.

This is true because $H\vec{J} = 0$ gives us the recursive relation on the coefficients P_d of $\vec{J}$.

Now we know that the relations in quantum cohomology are the symbols of differential operators which annihilate $\vec{J}$. How to find such operators? The answer is given by Kim's Lemma:

Lemma 2. *Suppose that*

(1) *Some differential operator $D(\hbar q\frac{\partial}{\partial q}, q, \hbar)$ commutes with H.*
(2) *$D\vec{J} = 0$ modulo q.*

Then $D\vec{J} = 0$.

In particular $D(p\circ, q, 0) = 0$ in $QH^*(X)$.

We have reduced the problem to the following question: how to find differential operators which commute with H and whose constant coefficient part is W-invariant? The way to solve this problem is to construct some quantum-mechanical system with the Hamiltonian H and to find the quantum conservation laws. This will be done in the next lecture.