TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 10

A. GIVENTAL

Recall the the system of differential equations

$$h\partial_\alpha S = \phi_\alpha \circ S \tag{1}$$

has a basis of solutions (we sum over repeated Greek indices)

$$S_{\mu\nu}\eta^{\mu\gamma}\phi_\gamma$$

defined by the bilinear form $S_{\mu\nu}$. It can be written invariantly as

$$\langle\phi|S|\psi\rangle := \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{d\in\Gamma} q^d(\phi, t, \ldots, t, \frac{\psi}{h-c})_{0,n+2,d}$$

where the undefined term $(n = 2, d = 0)$ is defined to be $\langle\phi,\psi\rangle$.

By Poincaré duality we can also view S as taking values in $H^*(X \times X)$, or think of S as a matrix. Remember that S depends on the variables t, q, and the parameter h.

Given S we can in principle recover the quantum multiplication $\phi_\alpha\circ$. In particular, we will begin to find a systematic way to derive relations in the quantum cohomology ring. This will allow us to eventually compute the (small) quantum cohomology ring of generalized flag manifolds (G/B).

In this lecture, we derive various properties of S. We see what the implication of the divisor equation is and we see how to restrict the differential equation 1 to $t = 0$ (for small Q.C.) and how the solution S restricts. It is in this setting that we are able to prove that operators that annihilate the first row of S give rise to relations in the small Q.C. ring.

If we write $t = t_0\mathbf{1}+t_1p_1+\cdots+t_rp_r+\sum_{\deg(\alpha)\geq 2} t_\alpha\phi_\alpha$ then intuitively, the divisor equation tells us that the dependence of S on t and q only involves $q_ie^{t_i}$ and t_α. We formalize this using auxiliary variables $\tau_1, \ldots, \tau_r$ with $\tau p := \tau_1p_1 + \cdots + \tau_rp_r$. We wish to compute

$$h\left(\frac{\partial}{\partial t_i} - q_i\frac{\partial}{\partial q_i}\right)S(t+\tau p, q)$$

and then apply the divisor equation. First recall that the effect of a t_i differentiation on S is to replace one of the t's (in this case one of the $t + \tau p$'s) by p_i. The effect of the $q_i\partial/\partial q_i$ differentiation on $q^d = q_1^{d_1}\cdots q_r^{d_r}$ is to multiply by d_i (recall that we write $d = d_1p_1 + \cdots d_rp_r$). Recall that the divisor equation when we have

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Notes taken by J. Bryan. I'm afraid this lecture is going be sketchier than my previous one due to time constraints.

gravitational descendants is

$$(T_1, \dots, T_n, p)_{g,n+1,d} = \langle p, d\rangle (T_1, \dots, T_n)_{g,n,d} + \sum_{i=1}^{n} (T_1, \dots, pDT_i, \dots, T_n)_{g,n,d}.$$

Finally, we will use the fact that

$$D(\frac{h}{h-c}) = \frac{1}{h-c}.$$

Combining these facts we can compute that

$$h\left(\frac{\partial}{\partial t_i} - q_i\frac{\partial}{\partial q_i}\right) S_{\mu\nu}(t+\tau p, q) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_d q^d(\phi_\mu, t+\tau p, \dots, \frac{p_i\phi_\nu}{h-c}).$$

We can write the above invariantly as

$$h\left(\frac{\partial}{\partial \tau} - q\frac{\partial}{\partial q}\right) S(t+\tau p, q) = S(t+\tau p, q) \cup (\mathbf{1}\otimes p) \tag{2}$$

where the cup product is in $H^*(X\times X)$.

Theorem 1. *The above formula implies that*

$$S_{\mu\nu}(t+\tau p, q) = S_{\mu\epsilon}(t, qe^\tau)\eta^{\epsilon\epsilon'}\langle e^{p\tau/h}\phi_{\epsilon'}, \phi_\nu\rangle$$

or, written more invariantly

$$S(t+\tau p, q) = S(t, qe^\tau) \cup (1\otimes e^{\tau p/h})$$

where the cup product is in $H^*(X\times X)$. We can also phrase this in terms of the bilinear form:

$$\langle\phi|S(t+\tau p, q)|\psi\rangle = \langle\phi|S(t, qe^\tau)|\psi e^{\tau p/h}\rangle.$$

Proof: This follows because the equations are true for $\tau = 0$ and both sides of the equation satisfy the same 1st order O.D.E., namely Equation 2.

Exercise. Define a bilinear form $V(t, q, x, y)$ by the formula

$$V_{\alpha\beta}(t, q, x, y) := \sum_{n=0}^{\infty}\frac{1}{n!}\sum_d q^d(\frac{\phi_a}{x-c}, t, \dots, t, \frac{\phi_\beta}{y-c})_{0,n+2,d}$$

with the $n = d = 0$ term defined by $\langle\phi_\alpha, \phi_\beta\rangle/(x+y)$. Prove that V satisfies

(1) $\partial_0 V_{\alpha\beta}(x, y) = \partial_0 S_{\epsilon\alpha}(x)\eta^{\epsilon\epsilon'}\partial_0 S_{\epsilon'\beta}(y)$. Hint: use WDVV.

(2) Derive from (1) that

$$S_{\epsilon\alpha}(h)\eta^{\epsilon\epsilon'}S_{\epsilon'\beta}(-h) = \eta_{\alpha\beta}.$$

(3) By applying the string equation to (1) prove that

$$V_{\alpha\beta} = \frac{1}{x+y}S_{\epsilon\alpha}(x)\eta^{\epsilon\epsilon'}S_{\epsilon'\beta}(y).$$

Remarks: The meaning of (2) is the following. While S satisfies $h\partial_\alpha S = \phi_\alpha \circ S$, (2) implies that the formal adjoint S^* satisfies $-h\partial_\alpha S^* = \phi_\alpha\circ S^*$. We also remark that *a priori* it is not clear that $1/(x+y)$ divides $S_{\epsilon\alpha}(x)\eta^{\epsilon\epsilon'}S_{\epsilon'\beta}(y)$ (thought of as power series in $1/x$ and $1/y$), but (3) tells us that is does.

Our system of O.D.E.'s lives on the space H which can be written

$$H = H^{*\ (\text{not } 2)}(X, \mathbb{C}) \oplus H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z}).$$

The small quantum cohomology ring restricts the parameter space H to the second factor in the above decomposition. The restricted system of O.D.E.'s is

$$hq_i\frac{\partial}{\partial q_i}S = p_i\hat{\circ}S$$

where $\hat{\circ}$ denotes the product in the small quantum cohomology ring:

$$\langle\phi_\alpha, p_i\hat{\circ}\phi_\beta\rangle := \sum_d q^d(\phi_\alpha, p_i, \phi_\beta)_{0,3,d}$$

We can find the fundamental solution to the restricted ODE essentially by restricting our fundamental solution to the unrestricted equations.

Theorem 2. *The fundamental solution to the equations* $hq_i\frac{\partial}{\partial q_i}S = p_i\hat{\circ}S$ *is determined by the matrix*[1]

$$S_{\mu\nu} = \sum_{d\in\Lambda} q^d(\phi_\mu, \frac{\phi_\nu e^{p\log q/h}}{h-c})_{0,2,d}$$

and the $n = d = 0$ *term is defined to be* $\langle\phi_\mu, \phi_\nu e^{p\log q/h}\rangle$.

Proof: We need to restrict our previous fundamental solution to the H^2 part of H. One way to restrict S to H^2 is to set $t = 0$ and $q = 1$ in the expression $S(t+\tau p, q)$ so that the variable τ parameterizes the second cohomology. From Theorem 1 we have

$$S_{\mu\nu}(t+\tau p, q)|_{t=0,q=1} = \sum_d e^{d\tau}(\phi_\mu, \frac{\phi_\nu e^{\tau p/h}}{h-c})_{0,2,d}.$$

Since we would rather parameterize the d_ip_i cohomology class with $q_i^{d_i}$ rather than $e^{d_ip_i}$, we can substitute into the above equation $\tau = \log q$ to get the theorem. Note that we should then think of q_i and $\log q_i$ as independent variables and assign them degrees $c_1(p_i)$ and 0 respectively in order to make S homogeneous. We could also prove the theorem directly, using the same argument that we used to show that the unrestricted fundamental solution is a solution (see previous lecture).

We extract the first *row* of S and show that operators that annihilate it give rise to relations in the small quantum cohomology ring. We define $J_\nu := S_{0\nu}$ or, more invariantly, J is characterized by:

$$\langle J, \phi\rangle = \langle\mathbf{1}|S|\phi\rangle.$$

Using the proposition and the string equation we can write an explicit formula for J:

$$J = e^{p\log q/h}\left(1 + \frac{1}{h}\sum_{d\neq 0} q^d ev_*(\frac{1}{h-c})\right)$$

where $ev : X_{0,1,d} \to X$ is the evaluation map.

Example. In the case where $X = \mathbb{CP}^1$, J is

$$e^{p\log q/h}\sum_{d=0}^{\infty}\frac{q^d}{(p+h)^2(p+2h)^2\cdots(p+dh)^2} \quad \text{mod } p^2.$$

[1] Warning: we use the same letter S to denote the fundamental solutions to the restricted equations and the unrestricted equations. It will hopefully clear from the context which S is which.

The importance of J is the following key lemma:

Lemma 1. *Suppose that*

$$D(hq_1\frac{\partial}{\partial q_1},\dots,hq_r\frac{\partial}{\partial q_r},q_1,\dots,q_r,h)$$

is an operator that is polynomial in the $hq\partial/\partial q$ variables and a power series in q and h, and suppose that $DJ=0$. Then

$$D(p_1\circ,\dots,p_r\circ,q_1,\dots,q_r,0)=0$$

is satisfied in $QH^*(X)$.

Remark:[2] If we consider $\mathcal{D}$, the space of differential operators on $H^2(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z})$ then those operators that annihilate J form an ideal $(\mathrm{Ann}\,J)$ and we can consider

$$\mathcal{D}J=\frac{\mathcal{D}}{(\mathrm{Ann}\,J)}$$

as a $\mathcal{D}$-module. From this point of view $QH^*(X)$ is the ring of functions on the character variety of the $\mathcal{D}$-module $\mathcal{D}J$. In some sense, QH should really be called "semi-classical cohomology" rather than "quantum cohomology".

We can illustrate the phenomenon of the Lemma in ordinary cohomology (although it is somewhat trivial). In this case it says that if $D(\frac{\partial}{\partial\tau})$ annihilates $e^{\tau p}$, then $D(p)=0$ in $H^*(X)$.

Proof of Lemma: We apply D to the whole matrix S. The hypothesis says that the first row of

$$DS=(D(p\circ,q,0)+o(h))S$$

vanishes. Since S is invertible the first row of $D(p\circ,q,0)+o(h)$ must also vanish. Said invariantly

$$\langle 1\circ D(p\circ,q,0),\phi\rangle=0 \mod h$$

for all ϕ and so $D(p\circ,q,0)=0$.

[2] I do my best to regurgitate the correct words of this remark. I don't know D-modules at all, so I'm not sure if I got it right or not. -J.B.