# TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
# LECTURE 8

A. GIVENTAL

## 1. Quantum Cohomology

The cohomology ring $H^*(X)$ has a cup product $\cup$, and $\langle a \cup b, c\rangle$ counts the number of points in the intersection of the cycles $a, b, c$. The idea of quantum cohomology is we can also count higher $\mathbb{C}\mathbf{P}^n$'s through the cycles. We define two quantum cohomology rings as follows.

**small QC ring:** $\langle a \circ b, c\rangle = \sum_{d\in\Lambda} \frac{1}{3!}(a, b, c)_{0,3,d} q^d = F_{\alpha\beta\gamma}(q, 0)$
**large QC ring:** $\langle \phi_\alpha \circ \phi_\beta, \phi_\gamma\rangle = F_{\alpha\beta\gamma}(q, t)$

These are associative and (super-)commutative $q$-deformations of the cup product with identity $\mathbf{1}$, graded by $\frac{1}{2}\deg\phi_\alpha$, $\deg q^d = \langle c_1(T_X), d\rangle$, and $\deg t_\alpha = 1 - \frac{1}{2}\deg\phi_\alpha$. As before, $q$ is a formal variable, but it actually has a natural interpretation as the transcendental coordinates on $H = H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$, i.e. the space of characters of $H_2(X, \mathbb{Z})$.

## 2. "Computing Quantum Cohomology"

There are two notions of what it means to compute the quantum cohomology ring of a manifold, and we'll illustrate this difference in the case of the Grassmanian $X = \mathbb{G}(k, k+m)$. (This is due to Witten.) First we'll find a presentation for the regular cohomology ring. Let $V$ be the rank $k$ tautological subbundle of $\mathbb{C}^{k+m}$ and let $W$ denote its rank $m$ complement $\mathbb{C}^{k+m}/V$. If we denote by $c_*$'s and $\tilde{c}_*$'s the Chern classes of $V$ and $W$, we have the equation

$$x^{k+m} = (x^k + c_1 x^{k-1} + \cdots + c_n)(x^m + \tilde{c}_1 x^{m-1} + \cdots + \tilde{c}_m),$$

and this induces relations on the $c$'s and $\tilde{c}$'s. In fact, $H^*(X)$ is the ring generated by the $c$'s and $\tilde{c}$'s with these relations. (Exercise.)

Now what about $QH^*(X)$? It should be a deformation of $H^*(X)$, and there is only one $q$ variable since $H^2(X)$ is one-dimensional, so

$$QH^*(X) \cong \mathbb{Q}[c_*, \tilde{c}_*, q]/(\text{deformed relations}).$$

Let's find the degree of $q$. To do this, we first need to find a generator of $H_2(X, \mathbb{Z})$, so consider the $V$'s with $\mathbb{C}^{k-1} \subset V \subset \mathbb{C}^{k+1} \subset \mathbb{C}^{k+m}$. These are parameterized by some $\mathbb{C}\mathbf{P}^1$ in $X$. Now observe that restricted to this $\mathbb{C}\mathbf{P}^1$, the bundle $W$ is an extension of $\mathbb{C}^{m-1}$ by $\mathcal{O}(1)$, so $\langle \tilde{c}_1, [\mathbb{C}\mathbf{P}^1]\rangle = 1$ and hence this class $[\mathbb{C}\mathbf{P}^1]$ can't be a non-trivial multiple of any other. The second ingredient in the formula for the degree of $q$ is the first Chern class of the tangent bundle. It is not hard to see that $T_X = V^* \otimes W$, so $c_1(T_X) = \text{rank}(V^*)c_1(W) + \text{rank}(W)c_1(V^*) = (k+m)\tilde{c}_1$ and $\deg q = \langle c_1(T_X), [\mathbb{C}\mathbf{P}^1]\rangle = k + m$.

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*Date*: September 18, 1997.
Notes taken by Jim Borger.

Now let's return to the defining relations. As the quantum cohomology ring is a deformation of the usual one, the relation in $H^*(X)$ given above must deform like

$$(x^k + c_1 x^{k-1} + \cdots + c_k) \circ (x^m + \tilde{c}_1 x^{m-1} + \cdots + \tilde{c}_m) = x^{k+m} + q \cdot ?.$$

But the degree of $q$ is $k+m$ and any relations must be homogeneous, so ? is a constant, i.e. some multiple of $\mathbf{1}$, and we can find this multiple by pairing it with the class $[pt.] \in H_0(X)$. Thus $q? = < c_k \circ \tilde{c}_m, [pt.] >$, so $? = (c_k, \tilde{c}_m, [pt.])_{0,3,1}$. To compute this, we will realize the Poincaré duals of $c_k$ and $\tilde{c}_m$ as generic cycles and compute the number of degree one rational curves through them and a generic point. Actually, we'll use $(-1)^k c_k = c_k(V^*)$ instead of $c_k$ since it's positive. So let $v$ be a generic section of $\mathbb{C}^{k+m}$. Its zero locus, when viewed as a section of $W$, is then $\{[V] \in X : v \in V\}$. Similarly, let $f$ be a generic section of $(\mathbb{C}^{k+m})^*$. Its zero locus, when viewed as a section of $V^*$ is $\{[V] \in X : f|_V = 0\}$. Finally, let $[V_0]$ be some generic point. Now we need to find the number of rational curves of degree one through these three cycles. But every rational curve of degree one can be constructed the way $[\mathbb{C}\mathbf{P}^1]$ was above. So we are reduced to computing the number of flags $E^{k-1} \subset F^{k+1}$ that have intermediate spaces $V_0, V_1, V_2$ such that

- (i) $V_0$ is the same $V_0$ as above,
- (ii) $v \in V_1$, and
- (iii) $f$ vanishes on $V_2$.

Since all our choices were generic, $F = \langle v \rangle + V_0$ and $E = \ker(f) \cap V_0$, so the flag is uniquely determined. Therefore

$$? = (c_k, \tilde{c}_m, [pt.])_{0,3,1} = (-1)^k (c_k(V^*), \tilde{c}_m, [pt.]) = (-1)^k.$$

For example, if $k = 1$ then $X = \mathbb{C}\mathbf{P}^m$ and $QH^* = \mathbb{Q}[p,q]/(p^{\circ(m+1)} - q)$.

The problem with such a presentation is that it gives no information about the classical product. For instance we can't calculate $\langle p^k \circ p^l, p^r \rangle$ on $\mathbb{C}\mathbf{P}^m$ from it. This brings us to the second notion of computing the quantum cohomology, and that's computing all the structural constants of $\circ$ with respect to some (linear) basis. Let's calculate these for $\mathbb{C}\mathbf{P}^m$.

Let $p^k$ be the class of a codimension $k$ linear subspace. Since

$$\deg F_{\alpha\beta\gamma} = \frac{1}{2}(\deg \phi_\alpha + \deg \phi_\beta + \deg \phi_\gamma) - 3 \dim_{\mathbb{C}} X,$$

we can see $\deg \langle p^k \circ p^l, p^r \rangle = k + l + r - m \le 2m$. But $\deg q = m+1$, so for reasons of degree,

$$\langle p^k \circ p^l, p^r \rangle = \begin{cases} ?q^0 & \text{if } k+l+r-m = 0 \\ ?q^1 & \text{if } k+l+r-m = m+1 \\ 0 & \text{otherwise} \end{cases}$$

Since the quantum product is a $q$-deformation of the usual one, we know the first ? is 1. For the second ?, observe that the union of all lines through generic codimension $k$ and $m$ planes is $(m-k)+(m-l)+1 = r$, so it meets the generic codimension $r$ plane once, and $? = (p^k \circ p^l, p^r)_{0,3,1} = 1$. Putting this all together,

$$\langle p^k \circ p^l, p^r \rangle = \begin{cases} 1 & \text{if } k+l+r-m = 0 \\ q & \text{if } k+l+r-m = m+1 \\ 0 & \text{otherwise} \end{cases}$$

So,

$$p^k \circ p^l = \begin{cases} p^{k+l} & \text{if } k+l \leq m \\ qp^{k+l-m-1} & \text{if } k+l > m \end{cases}$$

**Theorem 1.** *(Kontsevich-Manin) If $H^*(X)$ is generated as an algebra by $H^2(X)$, then the structural constants of $QH^*(X)$ determine the potential $F(t,q)$, i.e. small quantum cohomology determines large quantum cohomology.*

**Examples:**

(1) $\mathbb{C}\mathbf{P}^2$. See last lecture.

(2) A Calabi-Yau threefold (for example a quintic in $\mathbb{C}\mathbf{P}^4$). It has a non-vanishing global 3-form $\omega$, and $c_1(T_X) = 0$, so RR-$\dim(X_{0,0,d}) = \langle 0, d\rangle + 3 - 3 = 0$, so generically we expect isolated rational curves in each degree $d$. Call the number of them $n_d$. The only non-trivial structural constant will be $\langle p \circ p, p\rangle$ for dimensional reasons, and

$$\langle p \circ p, p\rangle = 5 + \sum_{d \geq 1} n_d d^3 \frac{q^d}{1-q^d}.$$

(This is not a trivial fact. The $(1-q^d)^{-1}$ comes from multiple covers.)

*Exercise.* Show the WDVV equation gives no information about the $n_d$'s.