TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 7

A. GIVENTAL

1. Notation

As usual, let X be a compact connected Kähler manifold. In order to simplify commutativity issues we will also assume $H^*(X) = H^{even}(X)$. We will also assume $H^*(X, \mathbb{Z})$ is free, $\{\phi_\alpha\}_\alpha$ will denote a basis for $H^*(X)$, and $\eta_{\alpha\beta}$ will represent the intersection number $< \phi_\alpha, \phi_\beta >$ of the Poincaré duals. We may also have occaison to use the inverse of the matrix $(\eta_{\alpha\beta})$. Its entries $\eta^{\beta\gamma}$ give the decomposition of the Poincaré dual of the diagonal in $X \times X$ in terms of the basis $\{\phi_\beta \otimes \phi_\gamma\}_{\beta,\gamma}$. The canonical generator of $H^0(X)$ will be written $\mathbf{1}$.

2. Genus zero potential

Exercise. $(t_1, ..., t_n)_{0,n,d}$ is morally the number of rational curves in X passing through $t_1, ..., t_n$. Show that all this information allows us to reconstruct the Gromov-Witten invariants.

In order to analyze the structure of these, we introduce the **genus zero potential**

$$F(t, q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_d q^d (t, ..., t)_{0,n,d}$$

for $t \in H^*(X)$, q a formal parameter, and d ranging over $H_2(X, \mathbb{Z})$, that is we consider $F(t)$ as an element of the formal group ring $\mathbb{Q}[[H_2(X, \mathbb{Z})]]$. Since we're assuming $H_2(X, \mathbb{Z})$ is free, say with generators $q_1, ...q_r$, then we could think of $F(t)$ as lying in $\mathbb{Q}[[q_1^{\pm 1}, ..., q_r^{\pm 1}]]$ but would then have a problem with multiplication, so let Λ be the semi-group generated by all d such that there exists an n with $[X_{0,n,d}] \neq 0$. For any such d, $\int_\Sigma \omega \geq 0$ where ω is a symplectic form on X. Furthermore, any such ω can be deformed in any direction, so Λ is contained in some semi-simplicial cone. So if we take $q_1, ..., q_r$ to be a basis for this cone, then $F \in \mathbb{Q}[[q_1, ..., q_r]]$. Finally, if we write $t = \sum t_\alpha \phi_\alpha$, then the coefficient of $t_{\alpha_1} \cdots t_{\alpha_n}$ in F is $(\phi_{\alpha_1}, ..., \phi_{\alpha_n})$.

Now let $p_1, ..., p_r$ be a basis of H^2 dual to $q_1, ..., q_r$ and write $t = (t_0) + (t_1 p_1 + \cdots t_r p_r) + (\sum_\alpha t_\alpha \phi_\alpha)$, where the terms are the parts of degree zero, two, and at least four.

Here are some facts which are essentially restatements of some material in previous lectures.

(1) **modulo q:** $F(t, 0) = \frac{1}{6} \int_X t \wedge t \wedge t$

Proof. $X_{0,n,0} = X \times \overline{\mathcal{M}}_{0,n}$ and $\int_{X \times \overline{\mathcal{M}}_{0,n}} t^{\wedge n} = 0$ unless $n = 3$. □

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Notes taken by Jim Borger.

(2) **The String Equation:** $\frac{\partial}{\partial t_0}F = \frac{1}{2} < t,t >$

Proof. By #1, $\frac{\partial}{\partial t_0}F(t,0) = \frac{1}{2} < t,t >$, so we need only show this partial vanishes for the higher q terms. So take $d > 0$ and write $t = t_0\mathbf{1} + t'$. Then $(t,...,t)_{0,n,d} = t_0(t,...,t,\mathbf{1})_{0,n,d} + (t,...,t,t')_{0,n,d}$. Since $d > 0$, what was previously called the string equation implies the first term is zero. Applying this repeatedly, we see $(t,...,t)_{0,n,d} = (t',...,t')_{0,n,d}$, but this does not depend on t_0, so its t_0 -partial is zero. □

(3) **The Divisor Equation:** $\frac{\partial}{\partial t_i}F = q_i\frac{\partial}{\partial q_i}F + \frac{1}{2}\int_X p_i \wedge t \wedge t$ for $i \geq 1$

Proof. Using #1 again, it is easy to check the q^0 terms agree. For $d > 0$, we can apply what used to be called the divisor equation (for *any* $n \geq 0$) to conclude that $\frac{\partial}{\partial t_i}(t,...,t)_{0,n+1,d} = d_i(n+1)(t,...,t)_{0,n,d}$. The divisor equation then follows. □

(4) **The WDVV Equation:** Put $F_{\alpha\beta\gamma} = \frac{\partial}{\partial t_\alpha}\frac{\partial}{\partial t_\beta}\frac{\partial}{\partial t_\gamma}F$. Then $\sum_{\epsilon,\epsilon'} F_{\alpha\beta\epsilon}\eta^{\epsilon\epsilon'}F_{\epsilon'\gamma\delta}$ is totally symmetric in $\alpha,\beta,\gamma,\delta$.

Proof. By the composition law,

$$\frac{1}{k!}\delta_{1234}(\phi_\alpha,\phi_\beta,\phi_\gamma,\phi_\delta,t,...,t)_{0,4+k,d} =$$
$$\sum \frac{1}{k'!k''!}(\phi_\alpha,\phi_\beta,t,...t,\phi_\epsilon)_{0,3+k',d'}\eta^{\epsilon\epsilon'}(\phi_{\epsilon'},t,...,t,\phi_\gamma,\phi_\delta)_{0,3+k'',d''}.$$

Since the left side is totally symmetric, so is the right. But

$$\sum_{\epsilon,\epsilon'} F_{\alpha\beta\epsilon}\eta_{\epsilon\epsilon'}F_{\epsilon'\gamma\delta} = \sum_{d,k} q^d RHS,$$

so the expression in question is totally symmetric. □

(5) **Grading:** Put $\deg t_\alpha = 1 - \frac{1}{2}\deg\phi_\alpha$ and $\deg q^d =< c_1(T_X), d >$ (so if $c_1(T_X) = \sum c^{(i)}p_i$, then $\deg q_i = c^{(i)}$). Then F is homogeneous of degree $3 - \dim_{\mathbb{C}} X$.

Proof. Consider the monomial

$$q^d t_{\alpha_1}\cdots t_{\alpha_n}\int_{[X_{0,n,d}]^{vir}} ev_1^*\phi_{\alpha_1}\wedge\cdots\wedge ev_n^*\phi_{\alpha_n}.$$

It's equal to 0 unless $\frac{1}{2}\sum\deg\phi_i = RR - \dim X_{0,n,d} =< c_1(T_X), d > + \dim_{\mathbb{C}} Xp + n - 3$, so all non-zero terms have degree $< c_1(T_X), d > + \sum(1 - \frac{1}{2}\deg\phi_i) = 3 - \dim_{\mathbb{C}} X$. □

Define the operation $\circ$ by $< \phi_\alpha \circ \phi_\beta, \phi_\gamma >= F_{\alpha\beta\gamma}$, i.e. $\phi_\alpha\circ\phi_\beta = \sum F_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}\phi_\gamma$.

Proposition 1. $\circ$ *is a symmetric, associative deformation of the cup-product with identity* $\mathbf{1}$.

Proof. It is symmetric because taking partials commutes. The other three are (once again) essentially restatements of the above properties. Here are two in detail.

Identity: $< \mathbf{1}\circ\phi_\beta, \phi_\gamma >= F_{\mathbf{1}\beta\gamma}$ which by the above form of the string equation is just $\frac{1}{2}\frac{\partial}{\partial t_\beta}\frac{\partial}{\partial t_\gamma} < t,t >=< \phi_\beta,\phi_\gamma >$ for all ϕ_γ so $\mathbf{1}\circ\phi_\beta = \phi_\beta$.

Deformation: $< \phi_\alpha\circ\phi_\beta, \phi_\gamma > |_{q=0} = F_{\alpha\beta\gamma}|_{q=0} = \frac{1}{6}\frac{\partial}{\partial t_\alpha}\frac{\partial}{\partial t_\beta}\frac{\partial}{\partial t_\gamma}\int_X t\wedge t\wedge t = \int_X \phi_\alpha\wedge\phi_\beta\wedge\phi_\gamma =< \phi_\alpha\cup\phi_\beta, \phi_\gamma >$ for all ϕ_γ, so $\phi_\alpha\circ\phi_\beta|_{q=0} = \phi_\alpha\cup\phi_\beta$. □

Remark. By the divisor equation, $F_{\alpha\beta\gamma}$ depends only on $q_1e^{t_1}, ..., q_re^{t_r}$, so the parameter space is really $H^*(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$, i.e. $\circ$ is formally a multiplication in each tangent space.

Examples:

(1) $X = \mathbb{C}\mathbf{P}^0 =$ a point:

$$(t\mathbf{1}, ..., t\mathbf{1})_{0,n,d} = \int_{\overline{\mathcal{M}}_{0,n,d}} (t\mathbf{1})^{\wedge n} = 0$$

unless $n = 3$ in which case it's t^3, so $F = \frac{1}{6}t^3$.

(2) $X = \mathbb{C}\mathbf{P}^1$: Let $p \in H^2(X, \mathbb{Z})$ denote the class of a hyperplane section and write $t = t_0\mathbf{1} + t_1 p$. Then F is a function of t_0, t_1, and q, and these have degrees 1,0, and 2. The q^0 term is simply $\frac{1}{6}\int_X < t, t, t >= \frac{1}{2}t_0^2 t_1$ so the rest is a function (divisible by q) of qe^{t_1} and t_0. Since F is homogeneous of degree $3 - \dim_{\mathbb{C}} X = 2$ and the degrees of q and t_0 are 2 and 1, there is some constant N such that $F = \frac{1}{2}t_0^2 t_1 + Nqe^{t_1}$. But N, being the coefficient of the q term in F, is just $(\ )_{0,0,1} = \int_{pt.} \mathbf{1} = 1$.

(3) $X = \mathbb{C}\mathbf{P}^2$: Write $t = t_0\mathbf{1} + t_1 p + \tau p^2$, where p is again the hyperplane class. The degrees of t_0, t_1, τ, and q are 1,0,-1, and 3, and the homogeneous degree of F is 1. As before, we can easily calculate the q^0 term; it's $\frac{1}{2}t_0 t_1^2 + \frac{1}{2}t_0^2\tau$. The string equation tells us that t_0 never occurs as a multiplier of t, so since F is homogeneous of degree 1,

$$F = \frac{1}{2}t_0 t_1^2 + \frac{1}{2}t_0^2\tau + \sum_{d>0}(qe^t)^d \tau^{3d-1}\frac{N_d}{(3d-1)!}$$

where $N_d = (p^2, ...p^2)_{0,3d-1,d} =$ the number of degree d rational curves passing through $3d-1$ generic points. (To see the expression for N_d, expand the exponential and look at the $q^d\tau^{3d-1}$ term.)

We know $N_1 = 1$, and, amazingly, with the help of the WDVV equation, this is all we need to calculate all the N_d's. Indeed, it is easy to see that $(\eta^{\epsilon\epsilon'})$ is the anti-diagonal matrix. Putting $\alpha = \beta = \tau$ and $\gamma = \delta = t_1$, the WDVV equation is

$$\begin{aligned} &F_{\tau\tau t_0}F_{\tau t_1 t_1} + F_{\tau\tau t_1}F_{t_1 t_1 t_1} + F_{\tau\tau\tau}F_{t_0 t_1 t_1} = \\ &\quad F_{\tau t_1 t_0}F_{\tau\tau t_1} + F_{\tau t_1 t_1}F_{t_1\tau t_1} + F_{\tau t_1\tau}F_{t_0\tau t_1}. \end{aligned}$$

Using the explicit representation we found for F, this simplifies to

$$F_{\tau\tau t_1}F_{t_1 t_1 t_1} + f_{\tau\tau\tau} = F_{\tau t_1 t_1}F_{t_1\tau t_1},$$

so $F_{\tau\tau\tau} = F_{\tau t_1 t_1}^2 - F_{\tau\tau t_1}F_{t_1 t_1 t_1}$. Now, considering the coefficient of $(qe^t)^d\tau^{3d-1}$ in both sides, we get

$$\frac{N_d}{(3d-4)!} = \sum_{d_1+d_2=d}\frac{N_{d_1}d_1^2}{(3d_1-2)!}\frac{N_{d_2}d_2^2}{(3d_2-2)!} - \frac{N_{d_1}d_1}{(3d_1-3)!}\frac{N_{d_2}d_2^3}{(3d_2-1)!}.$$

(Set $N_d = 0$ for non-positive d.) The first four values of N_d are $1, 1, 12$, and 620.