TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 6

A. GIVENTAL

1. Introduction

Today we shall discuss several examples of Gromov–Witten invariants, as well as some identities among them. We let X be a compact (almost-) Kähler manifold, and let $X_{g,n,d}$ denote the moduli space of stable degree d maps into X of genus g curves with n marked points (q.v. the notes from previous lectures).

Let us assume that we know what to make of $[X_{g,n,d}]$; as described last time, constructing the space $X_{g,n,d}$ and defining its virtual fundamental class are quite nontrivial tasks.

The Gromov–Witten invariants are introduced as follows: choose $(t_1, \dots, t_n)$, with $t_i \in H^*(X)$, pull them back to $X_{g,n,d}$, take the cup product, and evaluate over the fundamental class:

$$\begin{tikzcd} X_{g,n,d} \arrow[r, "\mathrm{ct}"] \arrow[d, "\mathrm{ev}_1,\dots,\mathrm{ev}_n"] & \overline{\mathcal{M}}_{g,n} \\ X & \end{tikzcd}$$

$$(t_1, \dots, t_n)_{g,n,d} := \int_{[X_{g,n,d}]} \mathrm{ev}_1^* t_1 \wedge \dots \wedge \mathrm{ev}_n^* t_n.$$

Interpretation. Number of genus g, degree d curves in X passing through the generic cycles Poincaré dual to $t_1, \dots, t_n$. [This is not precisely literally true, because of complicated transversality conditions that must be taken into account, but it is our interpretation.]

In addition, if $\alpha \in H^*\big(\overline{\mathcal{M}}_{g,n}\big)$, then

$$\alpha(t_1, \dots, t_n)_{g,n,d} := \int_{[X_{g,n,d}]} \mathrm{ct}^* \alpha \wedge \mathrm{ev}_1^*(t_1) \wedge \dots \wedge \mathrm{ev}_n^*(t_n),$$

which has the same interpretation, with the addition that $(\Sigma, \varepsilon_1, \dots, \varepsilon_n) \in \mathrm{PD}(\alpha)$ in $\overline{\mathcal{M}}_{g,n}$.

For example, let $\delta = \mathrm{PD}(\mathrm{pt})$ in $\overline{\mathcal{M}}_{g,n}$. Then $\delta(t_1, \dots, t_n)$ = number of maps $(\Sigma, \varepsilon) \xrightarrow{f} X$ such that $f(\varepsilon_i) \in \mathrm{PD}(t_i)$.

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Notes taken by Noam Shomron.

Example. Say we want to count the number of $(\mathbb{C}\mathbf{P}^1, z_1, \ldots, z_n) \xrightarrow{f} X$ with $f(z_i) \in t_i$. The total degree of the curve can be distributed between the components in any way, but it doesn't matter:

We get the same answer because of the definition of Gromov–Witten invariants (they depend only on the cohomology classes).

Automorphisms (of the maps f) do not play a part in this computation, since parametrized maps, by definition, have no automorphisms.

2. Gravitational descendants

They are defined as follows. Consider the universal stable map

$$\begin{tikzcd} X_{g,n+1,d} \arrow[r, "\mathrm{ev}_{n+1}"] \arrow[d, "\mathrm{ft}_{n+1}"] & X \\ X_{g,n,d} \arrow[u, bend left, "\varepsilon_1"] \arrow[u, bend left, "\cdots\ \varepsilon_n"'] & \end{tikzcd}$$

l_i = universal cotangent lines at the marked points

$c_i = c_1(l_i)$

(Here $l_i\big|_{[(\Sigma,\varepsilon)\to X]} = T^*_{\varepsilon_i}\Sigma$, and $c_1, \ldots, c_n$ are the 1st Chern classes of these.)

Definition 1. $\alpha(t_1 c_1^{d_1}, \ldots, t_n c_n^{d_n})_{g,n,d} = \displaystyle\int_{[X_{g,n,d}]} \mathrm{ct}^* \alpha \wedge \mathrm{ev}_1^* t_1 \wedge \cdots \wedge \mathrm{ev}_n^* t_n \; c_1^{d_1} \cdots c_n^{d_n}$ [the enumerative meaning of such invariants is subtle].

It is convenient to introduce the notation

Notation. $T(c) = t^{(0)} + t^{(1)} c + t^{(2)} c^2 + \cdots$, where $t^{(i)} \in H^*(X)$.

An example is given by the $t_i c^{d_i}$ in the above definition. With this notation, we can introduce the more general definition

Definition 2.

$$\alpha(T_1, \ldots, T_n)_{g,n,d} = \int_{[X_{g,n,d}]} \mathrm{ct}^* \alpha \wedge \bigwedge_{i=1}^{n} (\mathrm{ev}_i^* T_i)(c_i).$$

Next, we consider some identities among Gromov–Witten invariants.

3. The string equation

$$(T_1, \ldots, T_n, 1)_{g,n,d} = \sum_{i=1}^{n} (T_1, \ldots, DT_i, \ldots, T_n)_{g,n,d},$$

where $DT := \dfrac{T(c) - T(0)}{c}$ (shifts the sequence of coefficients by 1).

This is analogous to the string equation in Deligne–Mumford theory; the proof is the same as for $\overline{\mathcal{M}}_{g,n}$.

Notation. $\tilde{c}_i := c_i\big|_{X_{g,n+1,d}}$, $c_i := \mathrm{ft}_{n+1}^*\big(c_i\big|_{X_{g,n,d}}\big)$.

Compare $\tilde{c}_i$, $i \leq n$, and c_i. Then the relationship is $\tilde{c}_i = c_i + [\varepsilon_i]$. There is a geometric reason for the above relationship:

Note. These c_i's are not the same as $\mathrm{ct}^*\big(c_i\big|_{\overline{\mathcal{M}}_{g,n}}\big)$.

The following diagram commutes:

$$\begin{tikzcd}
X_{g,n+1,d} \arrow[d, "\mathrm{ft}_{n+1}"'] \arrow[dr, "\mathrm{ev}_1,\ldots,\mathrm{ev}_n"] & \\
X_{g,n,d} \arrow[r, "\mathrm{ev}_1,\ldots,\mathrm{ev}_n"'] & X
\end{tikzcd}$$

A particular case of the identity:

Corollary 1. $(t_1, \ldots, t_n, 1)_{g,n+1,d} = 0$

This has a clear geometrical meaning:

Remark. This is rigorous only when the foundation is established, i.e., $d = 0$, $g = 0$ and $n \leq 2$, or $d = 0$, $g = 1$, $n = 0$. There exists much confusion in the literature.

One more comment: we can put in any α, and have $\alpha(T_1, \dots, T_n, 1)$, but then α should be pulled back to $\overline{\mathcal{M}}_{g,n+1}$.

4. The dilation equation

$$(T_1, \dots, T_n, c_{n+1})_{g,n+1,d} = (2g - 2 + n)(T_1, \dots, T_n)_{g,n,d}$$

(c_{n+1} gets integrated over the fibre).

5. Divisor equation

Start with minimal generality: let $p \in H^2(X)$.

Theorem 1. $(t_1, \dots, t_n, p)_{g,n+1,d} = \langle p, d \rangle \, (t_1, \dots, t_n)_{g,n,d}$.

Proof. Pull back $X_{g,n+1} \xrightarrow{\mathrm{ev}_{n+1}} X$ and integrate over the fibre: get $\langle p, d \rangle$ (depends only on the homology class of the point). □

Enumerative meaning:

More generally,

$$(T_1, \dots, T_n, p)_{g,n+1,d} = \langle p, d \rangle \, (T_1, \dots, T_n)_{g,n,d} + \sum_{i=1}^{n} (T_1, \dots, pDT_i, \dots, T_n)_{g,n,d}.$$

6. WDVV equation (Composition law)

Let us begin with an example that shows the idea of this relation: say we want to count curves of genus 0 passing through 4 cycles, in a given configuration:

So we can count these different objects, which we consider as

Consider the diagonal $\Delta \subseteq X \times X$. Let $\{\phi_\alpha\}$ be a basis in $H^*(X)$; then $\{\phi_\alpha \otimes \phi_\beta\}$ is a basis of $H^*(X \times X)$. Let $\eta_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle$. Then $\mathrm{PD}(\Delta) = \sum_{\alpha,\beta} \eta^{\alpha\beta} \phi_\alpha \otimes \phi_\beta$ (this gives the diagonal constraint).

With this notation, what we are describing (for $\overline{\mathcal{M}}_{0,4}$) is

$$\delta(t_1, t_2, t_3, t_4)_{0,4,d} = \sum_{\alpha,\beta} \sum_{d'+d''=d} (t_1, t_2, \phi_\alpha) \eta^{\alpha\beta} (\phi_\beta, t_3, t_4).$$

The argument for respelling diagonal constraints like this is very general.

Proof. (scheme; we can make it rigorous in the convex case) Start with

$$\begin{array}{c} X_{0,4,d} \\ \downarrow \\ \overline{\mathcal{M}}_{0,4} \ni [\lambda] \end{array}$$

If $\lambda \to \infty$, the preimage is $X_{0,4,d}^{[\infty]} \subseteq X_{0,4,d}$

Splitting the curve into 2 parts gives

$$\coprod_{d'+d''=d} X_{0,3,d'} \times X_{0,3,d''} \xrightarrow{\mathrm{ev}'_{(\circ)} \times \mathrm{ev}''_{(\circ)}} X \times X$$

The preimage of Δ will make $f'(\circ) = f''(\circ)$. This gives

$$\begin{array}{c} \coprod_{d'+d''=d} X_{0,3,d'} \times_\Delta X_{0,3,d''} \\ \Big\downarrow \text{gluing map} \\ X_{0,4,d}^{[\infty]} \end{array}$$

The gluing map is not an isomorphism of varieties, but at a generic point it is a local isomorphism, since there is only one way to break up the curve. In any case, we will get an identification of the fundamental classes. ☐

7. Generalization

Describe some formulas arising from generalizations of the above: we can replace $X_{0,4,d}$ by $X_{0,4+k,d}$ (i.e., require that the first 4 points be in some fixed cross ratios). There will be several ways of dividing up the marked points between the components:

$$\delta_{1,2,3,4}(t_1, t_2, t_3, t_4, t, \ldots, t)_{0,4+k,d} =$$
$$\sum_{\alpha,\beta} \sum_{d'+d''=d} \sum_{k'+k''=k} \frac{k!}{k'!k''!}(t_1, t_2, t, \ldots, t, \phi_\alpha)_{0,k',d'} \eta^{\alpha\beta} (\phi_\beta, t, \ldots, t, t_3, t_4)_{0,k'',d''}.$$

Here $\delta \in \overline{\mathcal{M}}_{0,4+k}$ is the pullback $\pi^*(\delta)$ of $[\lambda] = \delta \in \overline{\mathcal{M}}_{0,4}$, where $\pi \colon \overline{\mathcal{M}}_{0,4+k} \to \overline{\mathcal{M}}_{0,4}$.

We can also put gravitational descendants in there:

$$\delta_{1,2,3,4}(T_1, \ldots, T_4, T, \ldots, T)_{0,4+k,d} = \text{as above.}$$

The idea behind all this is that $\overline{\mathcal{M}_{g,n}}^{\Gamma} \sim \coprod \overline{\mathcal{M}}_{g_\alpha, n_\alpha}$ [strata].

Let's try to explain what the general reduction formula looks like. Pick $\alpha_\Gamma = \mathrm{PD}\big[\overline{\mathcal{M}_{g,n}}^{\Gamma}\big]$; recall that Γ is represented by some graph, and if we cut an edge, the graph may or may not remain connected.

$$\alpha_\Gamma(T, \ldots, T)_{g,n,d} = \begin{cases} \sum_{\alpha,\beta} \alpha_{\tilde{\Gamma}}(T, \ldots, T, \phi_\alpha, \phi_\beta)_{g-1,n+2,d} \eta^{\alpha\beta} & \text{if } \tilde{\Gamma} \text{ is connected,} \\ \sum_{d'+d''=d} \alpha_{\Gamma_1}(T, \ldots, T, \phi_\alpha)_{g_1,k_1+1,d'} \eta^{\alpha\beta} \alpha_{\Gamma_2}(\phi_\beta, T, \ldots, T)_{g_2,k_2+1,d''} & \text{if } \tilde{\Gamma} = \Gamma_1 \coprod \Gamma_2 \end{cases}$$

(g_1 and g_2 correspond to Γ_1 and Γ_2).

We can start with any stratum and inductively knock out the edges of the graph, (recursively) producing a "composition law."

8. Next time

We continue, see what the WDVV equation means, and what structures it imposes.