TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURE 5

A. GIVENTAL

1. **Review**

Let us recapitulate what was going on last time. We let X be a compact Kähler manifold, and we described the moduli space $X_{g,n,d}$ of stable degree d maps $f\colon (\Sigma, \varepsilon) \to X$ of genus g curves with n marked points.

In order to define Gromov-Witten invariants, we need to make sense of the fundamental class $[X_{g,n,d}]$. In the genus 0 case, we had

Theorem (Behrend–Manin). *If $g = 0$ and X is convex*[1]*, then $X_{0,n,d}$ are compact complex orbifolds. If nonempty,* $\dim_{\mathbb{C}} X_{0,n,d} = \text{RR-}\dim_{\mathbb{C}} X_{g,n,d}$*,*

where $\text{RR-}\dim_{\mathbb{C}} X_{g,n,d}$ is defined as the "Riemann-Roch" dimension

$$\text{RR-}\dim_{\mathbb{C}} X_{g,n,d} = \langle c_1(TX), d\rangle + (1-g)(\dim_{\mathbb{C}} X - 3) + n.$$

Orbifolds have a natural fundamental cycle we can use. Unfortunately (or otherwise), in the general situation, the spaces $X_{g,n,d}$ can be very singular and unlike manifolds. In those cases, we want to define a virtual fundamental class $[X_{g,n,d}]^{\text{vir}}$.

2. **Gromov's idea for constructing symplectic invariants**

We want to construct invariants of symplectic manifolds. Gromov said: look at pairs (J, f), where $f\colon \mathbb{C}\mathbf{P}^1 \to (X, \omega)$ is a smooth map, and J is an ω-compatible almost complex structure. We are interested in the space of compatible pairs such that the map is J-holomorphic:

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[1] This means the tangent spaces are spanned by global holomorphic vector fields. Homogeneous spaces and tori are some examples.

The idea of the construction is that bordism invariants will give symplectic invariants, which do not depend on the choice of almost complex structure.

Approach.

(1) Pick a generic almost complex structure J.
(2) Pick generic constraints: require that $f(z_1), \ldots, f(z_N) \in$ given generic cycles, such that $\dim \mathcal{M}_{J,\text{cycles}} = 0$. The space $\mathcal{M}_{J,\text{cycles}}$ of solutions satisfying the constraints will therefore consist of a number of points. Since the manifold is 0-dimensional, the Gromov Compactness theorem implies that the number of points is finite.

$$\text{Define GW} = \sum_{s \in \mathcal{M}_{J,\text{cycles}}} \text{sign}(s).$$

(3) We want to show that we indeed have an invariant. Gromov's Compactness theorem tells us what maps we must add in order to compactify our space; the resulting space of maps contains some singular ones. We want to avoid encountering a "gluing" phenomenon

in our bordism. The singular maps will have $\operatorname{codim}_{\mathbb{C}} = 1 \implies \operatorname{codim}_{\mathbb{R}} = 2$, so they won't occur in dimension 1. Therefore we can go around them and produce a 1-parameter family

which describes an oriented bordism, and everything is okay.

This is the description of the traditional approach [Ruan, Ruan–Tian, ...]. It works well for $g = 0$ and ($c_1(TX) \geq 0$ or $\dim_{\mathbb{C}} X \leq 3$), but the approach fails beyond those assumptions, due to problems with multiple covers of holomorphic spheres with $\langle c_1(TX), d\rangle < 0$. [We are unable to bring everything to general position and achieve transversality, which breaks things at step 2.]

3. **Kontsevich's program**

The problem was solved by Kontsevich's program:

(1) Pick any J; take $\mathcal{M}_J$ (which may be very singular), compactify it by stable maps to $\overline{\mathcal{M}}_J$, and consider the resulting $X_{g,n,d}$.
(2) Then construct the virtual fundamental class $[X_{g,n,d}]^{\text{vir}}$.

The program was essentially completed by several authors.

4. **Model problem**

Suppose you have a vector bundle over a manifold, with a given section. If the section intersects the zero section transversely, then we can count the intersection number. Suppose it is not transverse to the zero section; then we still want to construct something representing the Euler class of the bundle.

There are two approaches:

(1) Deformation to the normal cone (algebro-geometric; q.v. Fulton)
(2) Topological — Ruan's approach to $[X_{g,n,d}]$:

Take some closed neighbourhood, and restrict the bundle to that neighbourhood. There exists a section not vanishing at the boundary $\implies$ Euler(bundle) $\in H^*(\text{nbhd}, \partial\text{nbhd})$ (recall that completely nonvanishing sections give zero). We want to define the virtual fundamental class via its pairing with cohomology classes; given some cohomology class t,

$$\int_{[X_{g,n,d}]^{\text{vir}}} t := \int_{[\text{nbhd}]} t \wedge \text{Euler(bundle)}.$$

5. **Examples**

(1) Illustrating the virtual fundamental class
(2) Difficulty of the traditional approach

Example 1. Consider $X_{1,1,0} = X \times \overline{\mathcal{M}}_{1,1}$. We know $\overline{\mathcal{M}}_{1,1} \cong \mathbb{C}\mathbf{P}^1$, so $\dim X_{1,1,0} = \dim_{\mathbb{C}} X + 1$, but

$$\text{RR-dim} = \langle c_1(TX), d\rangle + n + (1-g)(\dim X - 3) = 0 + 1 + 0 = 1.$$

Therefore the fundamental cycle is in the wrong dimension: 0, not 1. Where does transversality fail?

Linearize the Cauchy-Riemann equation:

$$0 \longrightarrow H^0(E, T_{f(0)}) \longrightarrow C^\infty(f^*T_{f(0)}) \xrightarrow{\overline{\partial}} \Omega^{0,1}(T_{f(0)}) \longrightarrow H^1(E, T_{f(0)}) \longrightarrow 0$$

Transversality would mean $\overline{\partial}$ is onto and 0 is a regular value, but the cokernel $\mathcal{H}^* \otimes T_{f(0)} := H^1(E, T_{f(0)})$ is nonzero, so it is not transversal.

Therefore, we have $X \times \overline{\mathcal{M}}_{1,1}$ plus a bundle

$$\begin{array}{c} \\ \big\downarrow{\scriptstyle \mathcal{H}^* \otimes TX} \\ X \times \overline{\mathcal{M}}_{1,1} \end{array}$$

so we get that the virtual fundamental class is $\mathrm{Euler}(\mathcal{H}^* \otimes TX)$.

Example of computation.

$$\int_{[X_{1,1,0}]} \mathrm{ev}_1^* t := \int_{X \times \overline{\mathcal{M}}_{1,1}} \mathrm{ev}_1^* t \ \mathrm{Euler}(\mathcal{H}^* \otimes TX),$$

where the second integral is over its fundamental class as an orbifold. Let $m = \dim_{\mathbb{C}} X$. Then

$$\mathrm{Euler}(\mathcal{H}^* \otimes TX) = c_m(TX) + c_{m-1}(TX)(-\omega) + c_{m-2}(TX)(-\omega)^2 + \cdots$$

[Euler class of a line bundle $\otimes$ a vector bundle]. Then $\omega^2 = 0$, so we are left with the two terms $c_m(TX) + c_{m-1}(TX)(-\omega)$. But $\int_{X \times \overline{\mathcal{M}}_{1,1}} c_m(TX) = 0$ (the dimension is wrong), so the first term vanishes, and we get

$$= \int_{[X \times \overline{\mathcal{M}}_{1,1}]} t \wedge c_{m-1}(TX)(-\omega) = -\frac{1}{24} \int_{[X]} c_{m-1}(TX) \wedge t.$$

Example 2. Here $X = \mathbb{C}\mathbf{P}^2$, and we study elliptic curves of degree 3 (cf. $X_{1,0,3}$). There are 10 degree 3 monomials: x^3, y^3, z^3, x^2y, x^2z, $\ldots$, $xyz \longrightarrow (\mathbb{C}\mathbf{P}^9)^*$, so the family of cubic elliptic curves in $\mathbb{C}\mathbf{P}^2$ has dimension 9.

Compare this with some of the maps in $X_{1,0,3}$:

Now,

(1) forms a 1-parameter family (in the closure of $(\mathbb{C}\mathbf{P}^9)^*$)

(2) $\dim = 9 \implies$ not in the closure of $(\mathbb{C}\mathbf{P}^9)^*$
(3) $\dim = 8 + 1 + 1 = 10 \implies$ not in the closure.

Worst of all, note that these 3 components all intersect each other, e.g., in

The problem with the traditional approach is that, in order to count objects, we must also consider some “pathological” objects, with a certain weight.

6. Next time

We try to describe some classes of Gromov-Witten invariants.