TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY
LECTURES 3-4

A. GIVENTAL

1. Calculating Universal Chern Numbers

1.1. **Witten's Conjecture = Kontsevich's Theorem.** We have seen that the Universal Chern Numbers satisfy some identities, including the String Equation

$$(T_1, \dots, T_n, 1)_{g,n+1} = \sum_{i=1}^{n} (T_1, \dots, DT_i, \dots, T_n)_{g,n}$$

where

$$DT = \frac{T(c) - T(0)}{c}$$

and the Dilation Equation

$$(T_1, \dots, T_n, c_{n+1})_{g,n+1} = (2g - 2 + n)(T_1, \dots, T_n)_{g,n}$$

These are special cases of Witten's Conjecture, proved by Kontsevich (Comm. Math. Phys. 147, pp. 1-23 (1992)).

Let

$$\mathcal{F}_g = \sum_{n=0}^{\infty} (T, \dots, T)_{g,n}/n!$$

Then there exist a collection of 2nd order differential operators in t_i

$$\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1, \dots$$

which commute like

$$\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, \dots$$

such that

$$\mathcal{L}_n \exp\left(\sum_{g=0}^{\infty} (\frac{2}{3})^{2-2g} \mathcal{F}_g\right) = 0$$

for all n. The statement for $n = -1$ reduces to the string equation, the statement for $n = 0$ to the dilation equation.

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Notes taken by D. Calegari.

1.2. **Product structure of Substrata.** $\overline{\mathcal{M}}_{g,n}$ is stratified by the combinatorial type of (Σ, ϵ). For a particular combinatorial type Γ, the substrata $\overline{\mathcal{M}}_{g,n}{}^{\Gamma}$ can often be decomposed, at least at the level of homology, as a product of simpler moduli spaces. Fo example:

so

$$\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{0,5}{}^{\Gamma}$$

is $1-1$ at generic points, and the computation of universal chern numbers can be pulled back to the product space.

A caveat is that one must be careful of orbifolds, in that the degree of these maps need not be 1.

1.3. **The Hodge Bundle.** On $\overline{\mathcal{M}}_{g,n}$ there is a bundle $\mathcal{H}$ whose fiber at (Σ, ϵ) is $H^0(\Sigma, \mathcal{K}_\Sigma)$, the global sections of the canonical bundle. By Serre duality, this is equal at each point to $H^1(\Sigma, \mathcal{O}_\Sigma)^*$ and is g-dimensional. This bundle is known as the *Hodge Bundle* of $\overline{\mathcal{M}}_{g,n}$.

Exercise.

(1) Prove that $\mathcal{K}_\Sigma$ is precisely the 1-forms on each irreducible component, which are allowed to have 1st order poles at the singular points, such that the sum of the residues on either component vanishes, for each intersection

(2)
$$l_{n+1} = K[\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n]$$

(3)
$$\mathcal{H}|_{\overline{\mathcal{M}}_{g,n}} = \mathrm{ft}^*\mathcal{H}|_{\overline{\mathcal{M}}_{g,n-1}} = \cdots = \mathrm{ft}^*\mathrm{ft}^* \ldots \mathrm{ft}^*\mathcal{H}|_{\overline{\mathcal{M}}_{g,0}}$$

(4) if ω denotes $c_1(\mathcal{H})$, compute

$$\int_{[\overline{\mathcal{M}}_{1,n}]} \omega^{d_0} c_1^{d_1} \ldots c_n^{d_n}$$

1.4. **The case $g = 1$.** For genus 1, we have $\overline{\mathcal{M}}_{1,0} = \emptyset$ and $\dim \overline{\mathcal{M}}_{1,n} = n$.

An elliptic curve with a marked point, together with a choice of generators of the fundamental group, is determined by a $\tau : \mathrm{Im}(\tau) > 0$. Then we have $E = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$.

Forgetting the choice of generators is tantamount to quotienting by the action of $PSL(2, \mathbb{Z})$, which acts on the upper half-plane by fractional linear transformations, and is generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. A fundamental domain for the action is

and the quotient space is $\mathbb{C}$, though as an orbifold it has two cone points of order 2 and 3 respectively.

$\overline{\mathcal{M}}_{1,1} = \mathbb{C}P^1$, and the curve corresponding to ∞ is given by the equation $y^2 = p_3(x)$, where p_3 is a cubic, two of whose roots coincide.

Recall that there is a fiber map

$$\mathrm{ft}_2 : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$$

The fiber over a point is the curve represented by that point modulo its automorphisms, which is generically $\mathbb{Z}_2$, but is $\mathbb{Z}_4$ and $\mathbb{Z}_6$ at special fibers. The fiber is therefore a $\mathbb{C}P^1$ almost everywhere, and at a few places counted with multiplicity.

Exercise. Show that $\overline{\mathcal{M}}_{1,2}$ can be described as ($\mathbb{C}P^2$ blown up at one point)$/S_3$ where S_3 is the group of automorphisms of $\mathbb{C}P^2$ permuting 3 generic lines passing through the blown-up point.

Let $c = c_1(\text{conormal bundle to}\epsilon_1)$ and $\omega = c_1(\mathcal{H})$. Then $c = \omega$, since a holomorphic differential on an elliptic curve is determined by its value at a marked point.

We make the claim

$$\int_{[\overline{\mathcal{M}}_{1,1}]} c = \int_{[\overline{\mathcal{M}}_{1,1}]} \omega = \frac{1}{24}$$

A factor of $1/6$ comes from the action of S_3. A factor of $1/2$ comes from the symmetry of the tangent bundle to the marked point. Finally, a factor of $1/2$ comes from the symmetry of a generic elliptic curve.

The justification comes from the dilation equation, which implies that

$$\int_{[\overline{\mathcal{M}}_{1,2}]} \omega c_2 = \int_{[\overline{\mathcal{M}}_{1,1}]} \omega$$

Geometrically, the integral is equal to

$$\int_{\Sigma/\mathbb{Z}_2} c_2 \int_{[\overline{\mathcal{M}}_{0,4}/S_3]} \omega = \frac{1}{2} \int_{[\overline{\mathcal{M}}_{0,4}/S_3]} \omega$$

Therefore ω on $[\overline{\mathcal{M}}_{1,1}]$ is $1/2$ that of the pullback of ω from $[\overline{\mathcal{M}}_{0,4}/S_3]$.

2. Moduli Spaces of Stable Maps

We want to study the moduli space of holomorphic maps $\Sigma \to X$ where X is a compact Kähler manifold, for example a subvariety of $\mathbb{C}P^n$, and such that the maps are *stable* in a sense to be made precise. We want to keep in mind degenerations of such maps of the form $xy = \epsilon$ as $\epsilon \to 0$ and $y^2 = x(x-1)(x-\lambda)$ as $\lambda \to 1$.

Definition 1. (1)

$$f : (\Sigma, \epsilon) \to X$$

is *holomorphic* if the restriction to each irreducible component is holomorphic. Here, as above, Σ is a compact connected curve with at most double singularities, and ϵ is a collection of ordered distinct non-singular marked points.

(2)

$$f' : (\Sigma', \epsilon') \to X, f'' : (\Sigma'', \epsilon'') \to X$$

are *equivalent* if there is an isomorphism from (Σ', ϵ') to (Σ'', ϵ'') commuting with f', f''.

(3) $f : (\Sigma, \epsilon) \to X$ is *stable* if it has no non-trivial infinitesimal automorphisms.

Example. • $f : E \to p \in X$ where E is an elliptic curve with no marked points is unstable.

• $f : (\Sigma, \epsilon) \to X$ where some component Σ_0 is a $\mathbb{C}P^1$ with < 3 special points, and $f|_{\Sigma_0}$ is constant, is unstable.

All other maps are stable.

(4) The *genus* of f, denoted $g(f)$, is defined to be $g(\Sigma)$.

(5) The *degree* of f is defined to be the homology class

$$\sum_{\alpha} f_*[\Sigma_\alpha] = d \in H_2(X, \mathbb{Z})$$

(6) $X_{g,n,d}$ is defined to be the set of equivalence classes of stable degree d maps to X of genus g curves with n marked points.

(7)

$$(ev_1, \ldots, ev_n) : X_{g,n,d} \to X \times \cdots \times X$$

is defined by evaluating the map f determined by a point in $X_{g,n,d}$ at its marked points.

(8)

$$ft_i : X_{g,n,d} \to X_{g,n-1,d}$$

is defined by forgetting the i-th marked point and contracting components which become thereby unstable.

(9) As with $\overline{\mathcal{M}}_{g,n}$ there are sections $\epsilon_i : X_{g,n,d} \to X_{g,n+1,d}$ and we have the universal stable map

$$\begin{tikzcd} ev_{n+1} : X_{g,n+1,d} \arrow[r] \arrow[d, "\epsilon_i"] & X \\ X_{g,n,d} & \end{tikzcd}$$

(10) There is another tautologically defined map

$$ct : X_{g,n,d} \to \overline{\mathcal{M}}_{g,n}$$

given by forgetting f and contracting components which become unstable.

Theorem 1. *(Gromov, Kontsevich, et. al.) $X_{g,n,d}$ has a natural structure of a compact Hausdorff topological space. In fact, it is a compact analytic orbifold. ev, ft, ct are continuous and analytic. The topology is defined by saying $f_n \to f$ if the images converge in the Hausdorff topology, and marked points converge.*

Example. (1)

$$X_{g,n,0} = X \times \overline{\mathcal{M}}_{g,n}$$

since degree 0 maps are constant.

(2)

$$(\mathbb{C}P^m)_{0,0,1} = Gr_{2,m+1}$$

which is already compact.

(3)

$$(\mathbb{C}P^1 \times \mathbb{C}P^1)_{0,0,(1,1)} = \text{compactification of space of graphs of automorphisms from } \mathbb{C}P^1 \to \mathbb{C}P^1$$

The space of such graphs is just $PSL(2, \mathbb{C})$, the represented as projective 2×2 complex matrices such that $ad \neq bc$. This can be compactified by removing the inequality, and we see that it naturally compactifies to $\mathbb{C}P^3$. The extra points correspond to maps whose images are the union of a vertical $\mathbb{C}P^1$ and a horizontal $\mathbb{C}P^1$.

(4)

$$(\mathbb{C}P^m)_{0,0,4}$$

has a number of components. For instance, a configuration of the kind

and a configuration of the kind

3. Gromov-Witten Invariants

Formally, the kind of invariants we would like to define are expressed as

$$\alpha(t_1, t_2, \ldots, t_n)_{g,n,d} = \int_{[X_{g,n,d}]} ct^*\alpha \wedge ev_1^* t_1 \wedge \cdots \wedge ev_n^* t_n$$

where $\alpha \in H^*(\overline{\mathcal{M}}_{g,n})$ and $t_i \in H^*(X)$.

Ideally, this expression should count the number of degree d genus g curves in X such that ev_i passes through the cycle t_i for all i, and (Σ, ϵ) range over the Poincaré dual of α.

Here α and t_i must be chosen to have the correct dimension, or the integral is defined to be zero.

Notice that by definition this depends only on $[\alpha]$, not the specific α, so we are free to choose a degenerate representative to make calculation easier.

The problem with making sense of this integral is deciding exactly what is meant by $[X_{g,n,d}]$.

Theorem 2. *(Behrend-Manin) If $g = 0$ and X is convex then $X_{0,n,d}$ is a compact complex orbifold. If nonempty, it has complex dimension*

$$dim_{\mathbb{C}} X_{0,n,d} = < c_1(TX), d > + dim_{\mathbb{C}} X + n - 3$$

Here X is *convex* if TX is spanned by global holomorphic vector fields. X is therefore a symmetric space, for example projective spaces, grassmanians, flag manifolds, etc.

4. Tangent Spaces to $X_{g,n,d}$

A point $p \in X_{g,n,d}$ is an equivalence class $[f]$ of maps $f : (\Sigma, \epsilon) \to X$. Some of the tangent space to $X_{g,n,d}$ is composed of variations of f keeping (Σ, ϵ) fixed. This is descibed by $H^0(\Sigma, f^*\mathcal{T}_X)$, where $\mathcal{T}$ is the tangent sheaf of X. $H^0(\Sigma, \mathcal{T}_\Sigma[-\epsilon])$, the infinitesimal automorphisms of (Σ, ϵ), injects into this by the assumption of stability. We must quotient by the image of this, since we are only interested of f up to equivalence.

We therefore have an exact sequence

$$0 \to H^0(\Sigma, \mathcal{T}_\Sigma[-\epsilon]) \to H^0(\Sigma, f^*\mathcal{T}_X) \to T_{[f]}X_{g,n,d}$$

The cokernel of this maps onto deformations of the complex structure on (Σ, ϵ), which are parameterized by $H^1(\Sigma, \mathcal{T}_\Sigma[-\epsilon]) \oplus_{s\in \mathrm{sing}(\Sigma)} T'_s \otimes T''_s$ but the image will miss $H^1(\Sigma, f^*\mathcal{T}_X)$. Finally, the cokernel is denoted by $\mathcal{N}[f]X_{g,n,d}$ which might be nontrivial because of the non-smoothness of X at some point.

Putting this together we have the exact sequence

$$\begin{gathered} 0 \to H^0(\Sigma, \mathcal{T}_\Sigma[-\epsilon]) \to H^0(\Sigma, f^*\mathcal{T}_X) \to T_{[f]}X_{g,n,d} \to \\ \to H^1(\Sigma, \mathcal{T}_\Sigma[-\epsilon]) \oplus T'_s \otimes T''_s \to H^1(\Sigma, f^*\mathcal{T}_X) \to \mathcal{N}_{[f]}X_{g,n,d} \to 0 \end{gathered}$$

Exercise. If $g = 0$ and X is convex, $H^1(\Sigma, f^*\mathcal{T}_X)$ is trivial.

If we define the dimension to be $\dim T - \dim N$ whenever N is trivial, then a computation shows that

$$\dim T - \dim N = \chi(f^*\mathcal{T}_X) - \chi(\mathcal{T}_\Sigma[-\epsilon]) + |sing(\Sigma)|$$

which can be computed via Riemann-Roch to be equal to $< c_1(TX), d > +(1 - g)(\dim_{\mathbb{C}} X - 3) + n$ We define this to be the *Riemann-Roch dimension* of $X_{g,n,d}$.

If $g = 0$ and X is convex, the Riemann-Roch dimension and the actual dimension coincide.

One can construct a class, the *virtual fundamental class* of $X_{g,n,d}$, denoted $[X_{g,n,d}]^{\mathrm{vir}} \in H_*(X_{g,n,d})$ of dimension equal to the Riemann-Roch dimension.

This result is due to (Behrend-Fantechi, Li-Tian) in the case X is algebraic, and to (Fukaya-Ono, Li-Tian, Ruan) when X is symplectic.

5. Symplectic Manifolds

A *Symplectic Manifold* is a manifold of even dimension X^{2m} together with a closed, non-degenerate (i.e. its highest exterior power is nowhere zero) 2-form ω.

By Darboux' theorem, any such manifold looks locally like $\mathbb{R}^{2m}$ with symplectic form $\sum_{i=1}^m dx_i \wedge dy_i$.

This theorem says that there are no local invariants of symplectic manifolds. In 1985 Gromov demonstrated that global invariants of symplectic structures could be found by studying solutions of the Cauchy-Riemann equations.

We can improve a symplectic structure to an *almost-Kähler structure*. That is, we can introduce an automorphism $J : TX \to TX$ such that $J^2 = -1$ such that J is compatible with ω in the sense that $\omega(J\xi, J\eta) = \omega(\xi, \eta)$, and $\omega(\xi, J\xi) > 0$ for $\xi \neq 0$.

Algebraically, this amounts to improving the $Sp(2n, \mathbb{R})$ structure on TX to a U_n structure, where TX is seen to be a $Sp(2n, \mathbb{R})$-bundle by Darboux' theorem.

If J is integrable, this gives a Kähler structure.

In general, J is not integrable, and we cannot look for complex submanifolds of X of dimension > 1. But we can find, at least locally, complex 1-submanifolds through any point, since the integrability condition is trivial in dimension 1.

We want to study the space of maps $f : \Sigma \to X_{\omega,J}$ such that $\bar{\partial}_J f = 0$.

Consider pairs J, f where $f : \mathbb{C}P^1 \to X$ and look at the subspace where $\bar{\partial}_J f = 0$. Notice that if we have flexibility in choosing J, this condition can be satisfied algebraically, so this subspace is a smooth infinite-dimensional submanifold. Let M_J be the fiber of this subspace over the projection map to J-space.

The space of compatible J's are contractible, since locally they are parameterized by $Sp(2n, \mathbb{R})/U_n$ which is contractible. More explicitly, there is an $Sp(2n, \mathbb{R})/U_n$ bundle over X, and a compatible J is a section of this.

Theorem 3. *(Compactification Theorem) If we have a sequence of maps $f_n : \Sigma_n \to X$ such that $\bar{\partial}_{J_n} f_n = 0$ and $J_n \to J$, and we assume that the area of $f_n(\Sigma_n)$ is bounded by some finite constant, then one can find a subsequence f_{n_k} which converge to a cusp curve $f_0 : \Sigma_0 \to X$ which is J_0-holomorphic, where Σ_0 is possibly singular.*

Notice that we need some control over the area of the maps, since the maps $z \to z^n$ from $\mathbb{C}P^1$ to itself converge nowhere.

We use the Riemannian metric defined by ω to determine area. But for a holomorphic curve, this area is the symplectic area, which depends only on the homology class of $f_*(\Sigma)$.