TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 2

A. GIVENTAL

1. Universal genus g curve with n marked points

Recall that we had the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1} & \longleftarrow & \dfrac{(\Sigma,\varepsilon)}{\mathrm{Aut}(\Sigma,\varepsilon)} \\ \Big\downarrow \underline{ft}_{n+1} & & \\ \overline{\mathcal{M}}_{g,n}, & & \end{array}$$

which we will continue to refer as "universal marked curves. Today we are going to do some intersection theory on $\overline{\mathcal{M}}_{g,n}$.

First, they are orbifolds. This implies the existence of fundamental class $[\overline{\mathcal{M}}_{g,n}]$. There are natural strata on them:

$\overline{\mathcal{M}}^{\Gamma}_{g,n} :=$ closure of {elements in $\overline{\mathcal{M}}_{g,n}$ with combinatorial structure specified by Γ}.

Example. Intersection of strata:

$$\begin{array}{c} \overline{\mathcal{M}}_{0,5} \\ \Big\downarrow \underline{ft}_{5} \\ \overline{\mathcal{M}}_{0,4} \end{array}$$

The strata a, b and c are specified by the following representing curves (rather than their dual graphs):

stratum a stratum b stratum c

(each stratum is isomorphic to $\mathbb{C}\mathbf{P}^1$.)

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These strata can be represented as a section of $\underline{ft}_5$ as:

Then we have

$$0 = (a+b)(a+b) = a^2 + 2ab + b^2 = a^2 + 2 + b^2,$$

where the first equality holds because $a+b$ is (homologous to) a fibre. (Fibres are homologous and they don't intersect each other.) The third equality comes from the fact that a intersects b geometrically at a point. Because the symmetry of a and b, we have $a^2 = -1$ and $b^2 = -1$. We also have $c^2 = -1$ by the combinatorial symmetry.

2. **Universal cotangent line**

Given the following diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\underline{ft}_{n+1}"] & \dfrac{(\Sigma,\varepsilon)}{\mathrm{Aut}(\Sigma,\varepsilon)} \arrow[l] \\ \overline{\mathcal{M}}_{g,n}. \end{tikzcd}$$

Definition 1. The universal cotangent line is defined to be the conormal (orbi-)bundle to the universal section ε_i, i.e.

$$l_i|_{[\Sigma,\varepsilon]} := T^*_{\varepsilon_i}\Sigma.$$

Notation. $c_i :=$ first chern class of l_i on $\overline{\mathcal{M}}_{g,n}$, $i = 1, \ldots, n$.
$\tilde{l}_i :=$ the corresponding universal cotangent line on $\overline{\mathcal{M}}_{g,n+1}$.
$\tilde{c}_i :=$ first chern class of $\tilde{l}_i$.

Definition 2.

$$(c_1^{d_1}, \ldots, c_n^{d_n})_{g,n} := \int_{[\overline{\mathcal{M}}_{g,n}]} c_1^{d_1} \cdots c_n^{d_n}.$$

Theorem 1.

$$(c_1^{d_1}, \dots, c_n^{d_n})_{0,n} = \begin{cases} \dfrac{(n-3)!}{d_1! \cdots d_n!} & \textit{if } d_1 + \cdots + d_n = n-3 \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

i.e.

$$\sum_{d_i \geq 0} x_1^{d_1} \cdots x_n^{d_n} (c_1^{d_1}, \dots, c_n^{d_n})_{0,n} = \int_{[\overline{\mathcal{M}}_{0,n}]} \prod_{i=1}^{n} \frac{1}{1 - x_i c_i} \tag{2}$$
$$= (x_1 + \ldots + x_n)^{n-3}$$

Lemma 1.

$$\int_{[\overline{\mathcal{M}}_{g,n+1}]} \tilde{c_1}^{d_1} \cdots \tilde{c_n}^{d_n} = \sum_{i, d_i \neq 0} \int_{[\overline{\mathcal{M}}_{g,n}]} c_i^{d_i - 1} \bigwedge_{j \neq i} c_j^{d_j}. \tag{3}$$

Proof. We will abuse notation to denote l_i and c_i also the line bundle and cohomology class induced from $\overline{\mathcal{M}}_{g,n}$, i. e. $l_i = \underline{ft}_{n+1}^*(l_i), c_i = \underline{ft}_{n+1}^*(c_i)$ on $\overline{\mathcal{M}}_{g,n+1}$. First we will examine the difference between l_i and $\tilde{l_i}$. They are almost the same except when the $(n+1)$-st point coincides with the i-th point, i. e. there exists a section s of $\tilde{l_i} \otimes l_i^{-1}$, $s \neq 0$ outside the divisor $[\varepsilon_i]$. ($[\varepsilon_i]$:= the image of universal section $\varepsilon_i(\overline{\mathcal{M}}_{g,n})$.) Thus we have

$$\tilde{c_i} = c_i + ?[\varepsilon_i].$$

To find the coefficient ? we notice that $l_i|_{[\varepsilon_i]} = l_i$ but $\tilde{l_i}|_{[\varepsilon_i]}$ is trivial. (Because 3 points on $\mathbb{C}\mathbf{P}^1$ give a coordinate and therefore trivialize the bundle.)

This implies $\tilde{l_i} \otimes l_i^{-1}|_{[\varepsilon_i]} = l_i^{-1}$. However, l_i^{-1} is exactly the normal bundle of $[\varepsilon_i]$, we have $? = 1$.

So $c_i^d = \tilde{c_i}^d + [-\varepsilon_i]^d$ because the intersection of $\tilde{c_i}$ and $[\varepsilon_i]$ is 0. Therefore

$$\begin{aligned} 0 &= \int_{[\overline{\mathcal{M}}_{g,n+1}]} \bigwedge_{i=1}^{n} c_i^{d_i} \\ &= \int_{[\overline{\mathcal{M}}_{g,n+1}]} \bigwedge (\tilde{c_i}^{d_i} [-\varepsilon_i]^{d_i}) \\ &= \int_{[\overline{\mathcal{M}}_{g,n+1}]} \bigwedge \tilde{c_i}^{d_i} + \int_{[\overline{\mathcal{M}}_{g,n+1}]} [-\varepsilon_i]^{d_i} \sum_{d_i > 0} \bigwedge_{j \neq i} c_j^{d_j} \\ &= \int_{[\overline{\mathcal{M}}_{g,n+1}]} \bigwedge \tilde{c_i}^{d_i} + \sum_{d_i > 0} \int_{[-\varepsilon]} [-\varepsilon_i]^{d_i - 1} \bigwedge_{j \neq i} c_j^{d_j} \end{aligned}$$

But $[\varepsilon_i] \cong \overline{\mathcal{M}}_{g,n}$ and $c_i = [-\varepsilon_i]$, we are done. $\square$

To deduce the theorem from the lemma we need some *combinatorics*. Let $T(c) := \sum_{j=0}^{\infty} t_j c^j$. Define

$$(T_1, \ldots, T_n)_{g,n} := \int_{[\overline{\mathcal{M}}_{g,n}]} T_1(c_1) \cdots T_n(c_n).$$

The lemma is equivalent to

$$(T_1, \ldots, T_n)_{g,n+1} = \sum_{i=1}^{n} (T_1, \ldots, DT_i, \ldots, T_n)_{g,n}\,,$$
$$DT(c) := \frac{T(c) - T(0)}{c}.$$

In particular,

$$\left(\frac{1}{1-x_1c}, \ldots, \frac{1}{1-x_nc}, 1\right)_{g,n+1} = (x_1 + \cdots + x_n)\left(\frac{1}{1-x_1c}, \ldots, \frac{1}{1-x_nc}\right)_{g,n}, \tag{4}$$

or more generally,

$$\begin{aligned}&\left(\frac{1}{1-x_1c}, \ldots, \frac{1}{1-x_nc}, 1, \ldots, 1\right)_{g,n+1} \\ =&(x_1 + \cdots + x_n)^k \left(\frac{1}{1-x_1c}, \ldots, \frac{1}{1-x_nc}\right)_{g,n}.\end{aligned} \tag{5}$$

Remark. These series of equations (3) (4) (5) are called *string equations.*

¿From now on, we will confine ourself to the case $g = 0$.

Proof. (of the theorem) We will apply induction on n (number of marked points). Notice that in the case $\overline{\mathcal{M}_{0,3}} \cong \mathrm{pt}$ the theorem is obvious. For general n:

$$\begin{aligned} 0 &= \int_{[\overline{\mathcal{M}}_{0,n}]} \bigwedge_{i=1}^{n} \left(\frac{1}{1-x_ic_i} - 1\right) \\ &= \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \int_{[\overline{\mathcal{M}}_{0,n}]} \bigwedge_{j \notin I} \frac{1}{1-x_jc_j} \\ &= \int_{[\overline{\mathcal{M}}_{0,n}]} \bigwedge_{i} \frac{1}{1-x_ic_i} + \sum_{I \neq \varnothing} (-1)^{|I|} \Big(\sum_{j \notin I} x_j\Big)^{|I|} \Big(\sum_{j \notin I} x_j\Big)^{n-3-|I|}. \end{aligned} \tag{6}$$

Here the first equality follow from the dimensional reason, while third equality from the above lemma and the induction hypothesis. Now observe that the second term of the last line in (6) is equal to

$$(-1)(x_1 + \cdots + x_n)^{n-3}$$

because it is the finite difference version of the equality

$$\frac{\partial^n (x_1 + \cdots x_n)^{n-3}}{\partial x_1 \cdots \partial x_n}\Big|_{\vec{x}=\vec{0}} = 0.$$

□

Exercise. Prove the following *"dilation equation"*.

$$(c_1^{d_1}, \ldots, c_n^{d_n}, c_{n+1})_{g,n+1} = (2g - 2 + n)(c_1^{d_1}, \ldots, c_n^{d_n})_{g,n}. \tag{7}$$

Hint:

$$c_{n+1} = K_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}} + [\varepsilon_1] + \cdots + [\varepsilon_n].$$