TOPICS IN ENUMERATIVE ALGEBRAIC GEOMETRY LECTURE 1

A. GIVENTAL

Outline of the course

1. Deligne-Mumford spaces
2. Moduli spaces of stable maps and GW-invariants
3. Quantum cohomology and Frobenius structure
4. Flag manifolds and Toda lattices
5. Singularity theory and the mirror conjecture
6. Equivariant cohomology and toric geometry
7. Equivariant GW-invariants
8. Elliptic GW-invariants

1. **Intorduction**

This course is to answer the question (partially) : How to count curves in Kähler manifolds?

Two main sources: Symplectic topology and String Theory. (Known as Gromov-Witten Invariants.)

Example. Classical examples of curve counting
(1) Generic two points in $\mathbb{C}\mathbf{P}^2$ determines a line.

Proof. A point in $\mathbb{C}\mathbf{P}^2$ determines a line in $\mathbb{C}\mathbf{P}^{2^*}$ and vice versa. The fact is true because 2 lines in $\mathbb{C}\mathbf{P}^{2^*}$ intersect at one point. □

(2) There is a unique quadratic curve passing through given five (generic) points.

Proof. A point in $\mathbb{C}\mathbf{P}^2$ determines a hyperplane in $\mathbb{C}\mathbf{P}^{5^*}$. Apply similar arguments. □

Remark. General comments about GW-invariants:
Gromov-Witten invariants are invariants of homotopy types of symplectic structures. The fact that there are many GW-invariants are good for symplectic geometors. On the other hand Witten showed that these invariants obey many universal relations, which is good for algebraic geometors. (These relations essentially come from the topology of $\overline{\mathcal{M}}_{g,n}$.)

There are two aspects of this topic:
1. How to use these relations for answering enumerative problems?
2. What is the algebraic structure of all GW-invariants?

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2. Deligne-Mumford spaces $\overline{\mathcal{M}}_{g,n}$

Notation. All curves considered in this course will be of the following type (called prestable curves).
Σ: compact connected complex curves with at most double singular points.
$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$: ordered, distinct, nonsingular marked points.

(Typical singularity : $xy = 0$.)

Definition 1. (1) $(\Sigma, \varepsilon) \leftrightarrows (\Sigma, \varepsilon)$ are equivalent if there is an isomorphism between two curves sending marked points to corresponding marked points.
(2) (Σ, ε) is stable if it has no nontrivial infinitesimal automorphism.

Example. Nonstable curves.
(1) $\mathbb{C}\mathbf{P}^1$ with less than three marked points.

(2) A curve has a rational component which has less than three special (marked + singular) points.

(3) Elliptic curves with no marked point.

All others are stable.

Definition 2. Arithmetic genus $g(\Sigma) := \dim H^1(\Sigma, \mathcal{O}_\Sigma)$.

Exercise.

$$(1)\quad g(\Sigma) = 1 - \frac{1}{2}\mathcal{X}(\Sigma \setminus sing(\Sigma))$$
$$= \sum_{\Sigma_\alpha:\text{irreducible}} g(\Sigma_\alpha) + \text{ number of cycles in the dual graph } \Gamma(=: h^1(\Gamma)).$$

Definition 3. A dual graph is a graph, where each vertex represents an irreducible component and each eadge represents a double point.

e.g.

curve dual graph

So

- $g = 0 \leftrightarrow \Sigma$ is a tree of $\mathbb{C}\mathbf{P}^1$'s.
- $g = 1 \leftrightarrow$ either Σ is a graph of $\mathbb{C}\mathbf{P}^1$'s with one cycle in the dual graph or Σ is a tree of cp^1's and one elliptic curves.

Definition 4. $\overline{\mathcal{M}}_{g,n}$: set of equivalent classes of stable genus g curves with n marked points.

Reference: Knudsen: *Proj. of* $\overline{\mathcal{M}}_{g,n}$, Math. Scand. 52 (1983).

Theorem 1. *$\overline{\mathcal{M}}_{g,n}$ has a natural structure of a compact complex orbifold.* [1]

Let us give a local description of the moduli space $\overline{\mathcal{M}}_{g,n}$. Recall that:
orbifold $\overset{\text{locally}}{=}$ manifold/finite group.

[1] i. e. a proper smooth stack.

manifold $\overset{\text{locally}}{=}$ tangent space.

The tangent space of the Deligne-Mumford moduli space can be described as:

$$0 \to H^1(\Sigma, \mathcal{T}_\Sigma[-\varepsilon]) \to T_{[\Sigma,\varepsilon]}\overline{\mathcal{M}}_{g,n} \to \oplus_{s\in \mathrm{sing}(\Sigma)} T'_s \otimes T''_s \to 0. \tag{2}$$

Exercise. Prove

$$H^1(\Sigma, \mathcal{T}_\Sigma[-\varepsilon]) = \oplus_{\Sigma_\alpha:\text{irreducible}} H^1(\Sigma_\alpha, \mathcal{T}_{\Sigma_\alpha}[-\text{special}].$$

Its dim $= 3g - 3 + n - \#(\text{singular points})$. Thus we have $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Definition 5. There are $n+1$ canonical morphisms:

$$\mathrm{ft}_i : \quad \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, \qquad i = 1, \ldots, n+1$$

which map an $(n+1)$-pointed curve to an n-pointed curve by forgetting the i-th point. These are called *forgetful maps*. Note that it might need to contract unstable components. For example

Question: What is $ft_{n+1}^{-1}(\mathrm{pt})$? ($\mathrm{pt}= [\Sigma, \varepsilon]$)
Answer: The fibre $= \Sigma/\mathrm{Aut}(\Sigma, \varepsilon)$.

So

$$\begin{array}{c} \overline{\mathcal{M}}_{g,n+1} \\ \big\downarrow \quad \scriptstyle{\varepsilon_i\text{:universal marked points}, \quad i=1\ldots n} \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

this can be viewed as the "universal (g, n) curve". [2]

Example. (1) $\overline{\mathcal{M}}_{0,n} = \varnothing$ for $n < 3$.
(2) $\overline{\mathcal{M}}_{0,3} = \mathrm{pt}$.

[2]Note that this terminology is not used in the usual sense of algebraic geometry.

(3) $\overline{\mathcal{M}}_{0,4} = \mathbb{C}\mathbf{P}^1$.

(4)

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,5} \arrow[d] & \arrow[l] \\ \mathbb{C}\mathbf{P}^1 & \end{tikzcd}$$

Exercise. Show that $\overline{\mathcal{M}}_{0,5} \cong \mathbb{C}\mathbf{P}^2$ blown up at 4 points.

Theorem 2. *(Kapranov) Choose $n-1$ generic points $q_1, q_2, \ldots, q_{n-1}$ in $\mathbb{C}\mathbf{P}^{n-3}$. The variety $\overline{\mathcal{M}}_{0,n}$ can be obtained from $\mathbb{C}\mathbf{P}^3$ by a series of blow ups of all the projective spaces spanned by q_i. The order of these bloww-ups can be taken as follows:*
(1) Points $q_1, \ldots, q_{n-2}$, and all the projective subspaces spanned by them (in order of increasing dimension, always).
(2) The point q_{n-1}, all the lines $< q_1, q_{n-1} >, \ldots, < q_{n-3}, \ldots, q_{n-1} >$ and subspaces spanned by them.
(3) The line $< q_{n-2}, q_{n-1} >$, the planes $< q_i, q_{n-2}, q_{n-1} >$, $i \neq (n-3)$ and all subspaces spanned by them.
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and so on.

Exercise. Prove that $\mathrm{Aut}(\Sigma, \varepsilon)$ are trivial for stable $g = 0$ marked curves.