Problem. *Compute*

$$\lim_{x\to 0}\frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

Solution. It is convenient to do the computation in the general form. Let

$$f(x) = x + ax^3 + bx^5 + cx^7 + o(x^7) \quad\text{and}\quad \phi(x) = x + \alpha x^3 + \beta x^5 + \gamma x^7 + o(x^7)$$

be the Taylor expanxions of two odd infinitely differentiable functions with the gerivatives at $x = 0$ equal to 1. Then

$$\begin{aligned} &f(\phi(x)) = \\ &\quad x + \alpha x^3 + \beta x^5 + \gamma x^7 + a(x + \alpha x^3 + \beta x^5)^3 + b(x + \alpha x^3)^5 + cx^7 + o(x^7) = \\ &\quad x + (a+\alpha)x^3 + (b + 3a\alpha + \beta)x^5 + (c + 3a\beta + 3a\alpha^2 + 5b\alpha + \gamma)x^7 + o(x^7). \end{aligned}$$

Since the coefficients at x^3 and x^5 are symmetric with respect to exchanging Latin and Greek letters, the expansion of $f(\phi(x)) - \phi(f(x))$ starts only with x^7. This is a bad news, but the good news is that the coefficient at x^7 depends on the Taylor coefficients of f and ϕ only up to order x^5:

$$f(\phi(x)) - \phi(f(x)) = [3a\alpha(\alpha - a) + 2(b\alpha - \beta a)]\, x^7 + o(x^7).$$

On the other hand, if $\phi = f^{-1}(x) = x + Ax^3 + Bx^5 + o(x^5)$, then

$$x = f(f^{-1}(x)) = x + (a + A)x^3 + (b + aA + B)x^5 + o(x^5),$$

i.e. $A = -a$, and $B = 3a^2 - b$. Substituting into $3a\alpha(\alpha - a) + 2(b\alpha - \beta a)$ respectively: $-a$ for a, $-\alpha$ for α, $3a^2 - b$ for b, and $3\alpha^2 - \beta$ for β, we find:

$$\begin{aligned} &f^{-1}(\phi^{-1}(x)) - \phi^{-1}(f^{-1}(x)) = \\ &\qquad \left[3a\alpha(a - \alpha) + 2(a(3\alpha^2 - \beta) - \alpha(3a^2 - b))\right] x^7 + o(x^5) = \\ &\qquad - [3a\alpha(\alpha - a) + 2(b\alpha - \beta a)]\, x^7 + o(x^7). \end{aligned}$$

Thus

$$\lim_{x\to 0}\frac{f(\phi(x)) - \phi(f(x))}{f^{-1}(\phi^{-1}(x)) - \phi^{-1}(f^{-1}(x))} = 1.$$

Could you explain (predict) this result without much computation?