Hints

8. $\overrightarrow{AA'} = 3\overrightarrow{AM}/2$.

14. $\cos^2\theta \le 1$.

19. Rotate the regular n-gon through $2\pi/n$.

22. Take $X = A$ first.

23. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

24. If $a+b+c+d=0$, then $2(b+c)(c+d) = a^2 - b^2 + c^2 - d^2$.

25. $XA^2 = |\overrightarrow{OX} - \overrightarrow{OA}|^2 = 2R^2 - 2\langle \overrightarrow{OX}, \overrightarrow{OA}\rangle$ if O is the center.

26. Consider projections of the vertices to any line.

27. Project faces to an arbitrary plane and compute signed areas.

35. Compute $(\cos\theta + i\sin\theta)^3$.

37. Divide P by $z - z_0$ with a remainder.

40. $(z - z_0)(z - \bar{z}_0)$ is real.

41. Apply Vieta's formula.

45. $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

46. See Figure 13.

50. Show first that $\binom{n}{k}$ equals the number of k-element subsets in the set of n objects.

52. Show that the sum of distances to the foci of an ellipse from points on a tangent line is minimal at the tangency point; then reflect the source of light about the line.

54. $e = |P'P|/|P'Q|$ is the slope of the secting plane.

55. Apply the forthcoming classification of quadratic curves.

56. $AX^2 + CY^2$ is even in X and Y.

57. $Q(-x, y) = Q(x, y)$ only if $b = 0$.

60. $\cos\theta = 4/5$.

62. Rotating the coordinate system as in Example on p. 11, don't forget to apply the change to the linear terms.

63. $Q(-x, -y) = Q(x, y)$ for a quadratic form Q.

65. Reflect the plane about a suitable line.

68. $x^2 + 4xy = (x + 2y)^2 - 4y^2$.

69. Examine when the quadratic equation $at^2 + 2bt + c = 0$ has a double root.

76. Given that A and B can be transformed into C, find a way to transform A into B.

80. For each of the normal forms, draw the region of the plane where the quadratic form is positive.

81. Draw the region on the plane where $xy > 0$.

85. First apply the Inertia Theorem to make $Q = x_1^2 + x_2^2$.

91. Take $x_1 = x,\ x_2 = \dot{x}$.

97. In the "toy version," one of the quadratic forms is $x^2 + y^2$, the squared Euclidean distance to the origin.

98. $4ac = (a + c)^2 - (a - c)^2$.

102. $\frac{3n(n+1)}{2} - n^2 = \frac{n(n+3)}{2} > \frac{n^2}{2}$.

103. $1 \neq 0$.

105. If $ba = 1$ and $ab' = 1$, then $b = bab' = b'$.

110. This is a rephrasing of the previous exercise.

114. If $(a, n) = 1$, then $ka + ln = 1$ for some integers k and l.

116. Show that a non-zero homomorphism between two fields maps 1 to 1, and inverses to inverses.

118. Construct the homomorphism $f : \mathbb{Z} \to \mathbb{K}$ such that $f(1) = 1$.

119. Take $\mathbb{F} = \{0, 1, \eta, \zeta\}$, and set $1 + 1 = 0$, $\eta + \zeta = 1$, $\eta\zeta = 1$.

120. $x^2 + x + 1 = (x + \eta)(x + \zeta)$.

121. Compute $\mathbf{0} + \mathbf{0}'$.

122. $-\mathbf{u} = (-1)\mathbf{u}$.

123. Redefine multiplication by scalars as $\lambda\mathbf{u} = \mathbf{0}$ for all $\lambda \in \mathbb{K}$ and all $\mathbf{u} \in \mathcal{V}$.

124. $0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$.

130. Rotated parallelograms are still parallelograms.

133. A linear function on $\mathcal{V} \oplus \mathcal{W}$ is uniquely determined by its restrictions to $\mathcal{V}$ and $\mathcal{W}$.

136. Translate given linear subspaces by any vector in the intersection of the affine ones.

142. Examine the kernel of the map $\mathbb{R}[x] \to \mathbb{R}^2$ which associates to a polynomial P the pair $(P(1), P(-1))$ of its values at $x = \pm 1$.

153. $\mathbf{cos}\, 4\theta + i\, \mathbf{sin}\, 4\theta = (\mathbf{cos}\, \theta + i\, \mathbf{sin}\, \theta)^4$.

155. Prove that E is injective by identifying $\mathcal{V}$ with $\mathbb{K}^n$ and evaluating coordinate functions on a non-zero vector.

159. To a linear form $f : \mathcal{V}/\mathcal{W} \to \mathbb{K}$, associate $\mathcal{V} \xrightarrow{\pi} \mathcal{V}/\mathcal{W} \xrightarrow{f} \mathbb{K}$.

161. $|\mathbb{F}| = p^n$, where n is the dimension of $\mathbb{F}$ as a $\mathbb{Z}_p$-vector space.

168. Both associate to an input $x \in X$ the output $w = f(g(h(x)))$.

177. Pick two 2×2-matrices at random.
180. Compute BAC, where B and C are two inverses of A.
181. Given two isomorphisms $B : \mathcal{U} \to \mathcal{V}$ and $A : \mathcal{V} \to \mathcal{W}$, find an isomorphism from $\mathcal{W}$ to $\mathcal{U}$.
182. Consider 1×2 and 2×1 matrices.
184. Solve $D^{-1}AC = I$ for D and C.
188. Which 2×2-matrices are anti-symmetric?
192. Start with any non-square matrix.
202. Each elementary product contains a zero factor.
205. There are 12 even and 12 odd permutations.
207. Permutation $\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$ has length $\binom{n}{2}$.
209. Pairs of indices inverted by σ and σ^{-1} are the same.
210. The transformation: *logarithm* $\to$ *lagorithm* $\to$ *algorithm* consists of two transpositions.
213. Locate a pair $(i, i+1)$ of nearby indices which σ inverses, compose σ with the transposition of these indices, and show that the length decreases by 1.
218. $247 = 2 \cdot 100 + 4 \cdot 10 + 7$.
222. $\det(-A) = (-1)^n \det A$.
224. $\det(C^t QC) = (\det Q)(\det C)^2$.
235. Apply the 1st of the "cool formulas."
236. Compute the determinant of a 4×4-matrix the last two of whose rows repeat the first two.
237. Apply Binet–Cauchy's formula to the 2×3 matrix whose rows are $\mathbf{x}^t$ and $\mathbf{y}^t$.
238. Show that $\Delta_n = a_n \Delta_{n-1} + \Delta_{n-2}$.
240. Use the defining property of **Pascal's triangle**: $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$.
241. Use the fact of algebra that a polynomial in $(x_1, \dots, x_n)$, which vanishes when $x_i = x_j$, is divisible by $x_i - x_j$.
242. Divide the kth column by k and apply Vandermonde's identity.
243. Product $p(x)q(y)$ of polynomials is linear in p and q.
244. $p(s)q(s')$.
246. (a) Product of linear forms is a bilinear form, and factors through $\mathcal{V} \otimes \mathcal{W}$; (b) $\mathbf{f} \otimes \mathbf{w}$ corresponds to the rank-1 linear map $\mathbf{v} \mapsto \mathbf{f}(\mathbf{v})\mathbf{w}$.
249. Use universality. E.g., consider the bilinear map $\mathcal{V} \times \mathcal{W} \to \mathcal{W} \times \mathcal{V} : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{w} \otimes \mathbf{v}$.
251. Use the operation of **symmetrization** $\mathcal{V}^{\otimes p} \ni \mathbf{a} \mapsto \frac{1}{p!} \sum_\sigma \sigma(\mathbf{a})$.
252. Polynomial functions on the Cartesian product of two spaces form the tensor product of the spaces of polynomial functions on the factors.

253. $\mathbf{e}_{i_1} \otimes \cdots \otimes e_{i_p} \in B_p$ if any of the indices i_k coincide.

257. Use universality.

265. The exterior $2n$-form $\mathbf{f}^{\wedge n}$ does not change under changes of coordinates.

268. Vectors and covectors are transformed differently.

271. $H : \mathbb{R}^7 \to \mathbb{R}^1$.

272. The space spanned by columns of A and B contains columns of $A + B$.

280. $(A^t\mathbf{a})(\mathbf{x}) = \mathbf{a}(A\mathbf{x})$ for all $\mathbf{x} \in \mathbb{K}^n$ and $\mathbf{a} \in (\mathbb{K}^m)^*$.

295. Modify the Gaussian elimination algorithm of Section 2 by permuting unknowns (instead of equations).

296. Apply the LUP decomposition to M^t.

297. Apply LPU, LUP, and PLU decompositions to M^{-1}.

302. When $\mathbb{K}$ has q elements, each cell of dimension l has q^l elements.

306. Consider the map $\mathbb{R}^n \to \mathbb{R}^{p+q}$ defined by the linear forms.

319. $T = A + iB$ where both A and B are Hermitian.

320. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{z}^\dagger \mathbf{w}$.

322. $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z}^\dagger \mathbf{w}$.

323. Use the previous exercise.

325. The normal forms are: $i|Z_1|^2 - i|Z_2|^2,\ |Z_1|^2 - |Z_2|^2,\ |Z_1|^2$.

330. Take the linear form for one of new coordinates.

332. Prove that -1 is a non-square.

335. There are 1 even and 3 odd non-degenerate forms.

339. $\frac{1}{3} = 1 - \frac{2}{3}$.

340. Start with inverting 2-*adic units* $\cdots * * * 1$. ($*$ is a wild card).

342. Use the previous exercise.

366. Apply Vieta's formula.

368. Compute $\langle \mathbf{x}, A\mathbf{x} \rangle$ for $\mathbf{x} = \sum t_i \mathbf{v}_i$, where $\{\mathbf{v}_i\}$ is an orthonormal basis of eigenvectors, and $\sum t_i^2 = 1$.

369. If $AB = BA$, then eigenspaces of A are B-invariant.

375. Find an eigenvector, and show that its orthogonal complement is invariant.

379. Construct $v_1, \ldots, \mathbf{v}_n$ by applying the Spectral Theorem to the non-negative Hermitian operator $A^\dagger A$, and check that $A\mathbf{v}_i \in \mathcal{W}$ corresponding to distinct nonzero eigenvalues μ_i of $A^\dagger A$ are pairwiseperpendicular.

380. Rephrase the previous exercise in matrix form.

383. The equation $\langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = 0$ quadratic in t has no real solutions unless $\mathbf{x}$ is proportional to $\mathbf{y}$.

385. $\langle \mathbf{x}, \mathbf{x} \rangle = (\sum x_i)^2 + \sum x_i^2$.

387. $\det(U^t U) = 1$.

396. See Example 1.

406. Use the previous exercise.

408. Prove first that the radius of the circle must be equal to the middle semiaxis of the ellipsoid.

411. For "exactly onne" claim, use the coordinate-less description of the spectrum provided by the minimax principle.

412. When $x \approx 0$, $\mathbf{sin}\, x \approx x$.

415. If T denotes the cyclic shift of coordinates in $\mathbb{C}^n$, then $C = C_0 I + C_1 T + \cdots + C_{n-1} T^{n-1}$.

428. Use Newton's binomial formula.

441. Pick A and B so that $e^A e^B \neq e^B e^A$.

444. Verify it first for a Jordan cell.

449. Make two of the lines the coordinate axes on the plane, and the third one the graph of an isomorphism between them.

450. If the first three lines are $y = 0$, $x = 0$, and $y = x$., then the fourth one is $y = \lambda x$.

451. If $d = 0$, $a = \infty$, $b = 1$, then $\lambda = c$.

453. Assuming that the spaces $\mathcal{U}_i$ have the same dimension, and the maps between them are invertible, consider the determinant (or eigenvalues) of the composition of the maps around the cycle.

456. Use the theory of Bruhat cells.

457. Use Lemma, and apply induction on the length of the shortest of the legs.

458. Use Sylvester's rule together with the previous exercise.

461. On the unit sphere of area 4π, the area of a triangle with the angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ has the area $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi$. Use this to compute the number of required triangles.

463. Compute $R_{\mathbf{v}_1} R_{\mathbf{v}_2}$ to show that it is a Jordan cell of size 2 with the eigenvalue 1.

464. They correspond to positive roots $\mathbf{e}_i - \mathbf{e}_j = \mathbf{v}_{i+1} + \cdots + \mathbf{v}_j$, where $0 \leq i < j \leq n$.

Answers

1. $mg/2$, $mg\sqrt{3}/2$.

2. 18 *min* (reloading time excluded).

6. It rotates with the same angular velocity along a circle centered at the barycenter of the triangle formed by the centers of the given circles.

10. $3/4$.

13. $7\overrightarrow{OA} = \overrightarrow{OA'} + 2\overrightarrow{OB'} + 4\overrightarrow{OC'}$ for any O.

15. $3/2$.

17. $2\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u} + \mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2$.

18. (b) No.

28. Yes; Yes (0); Yes.

29. (a) $\frac{1+5i}{13}$; (b) $\frac{1-i\sqrt{3}}{2}$.

31. $|z|^{-2}$.

33. (a) $\sqrt{2}, -\pi/4$; (b) $2, -\pi/3$.

34. $\frac{-1+i\sqrt{3}}{2}$.

38. $2 \pm i$; $i\frac{1\pm\sqrt{5}}{2}$; $1+i,\ 1+i$; $1 \pm \frac{\sqrt{6}-i\sqrt{2}}{2}$.

39. $-2, 1 \pm i\sqrt{3}$; $i, \frac{\pm\sqrt{3}-i}{2}$; $\pm i\sqrt{2}, \pm i\sqrt{2}$; $\frac{\sqrt{3}\pm i}{2}, \frac{\sqrt{3}\pm i}{2}$; $\pm i,\ \pm\frac{\sqrt{3}\pm i}{2}$.

42. $a_k = (-1)^k \sum_{1\le i_1<\cdots<i_k\le n} z_{i_1} \cdots z_{i_k}$.

45. $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$.

47. The upper halp-plane $\operatorname{Im} w > 0$.

49. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

58. Yes, $k(x^2 + y^2)$.

59. $y = \pm x$; level curves of (a) are ellipses, of (c) hyperbolas.

60. $2X^2 - Y^2 = 1$.

62. Semiaxes $2/\sqrt{3}$ and 2.

64. (a) $\pm(\sqrt{\alpha^2 - \beta^2}, 0)$, (b) $\pm(\sqrt{\alpha^2 + \beta^2}, 0)$.

68. 2nd ellipse, 1st & 4th hyperbolas.

72. A pair of intersecting lines, $x - 1 = \pm(2y - 1)$.

79. For normal forms, X^2 and 0 can be taken.

84. $(n+1)(n+2)/2$.

88. $x(t) = x(0)e^{\lambda t}$.

89. $x(t) = x(0)e^{3t}, y(t) = y(0)e^{-t}, z(t) = z(0)$.

92. $\dot{X}_1 = iX_1, \quad \dot{X}_2 = -iX_2$.

96. Yes, some non-equivalent equations represent the empty set.

101. $\frac{n(n+1)}{2}$.

107. Use the **Euclidean algorithm**: divide a by b with the remainder r, then divide b by r with the remainder r_1, then divide r by r_1 with the remainder r_2, etc. until the remainder becomes 0.

108. Use the previous exercises.

111. n.

113. $1, 3, 5, 7$.

135. Points, lines, and planes.

138. p^n; p^{mn}.

139. $p^{n(n-1)/2}$.

141. (b) $\operatorname{Ker} D = \mathbb{K}[x^p] \subset \mathbb{K}[x]$.

150. $\dim = 2$.

152. $x^k = x_1^k L_1(x) + \cdots + x_n^k L_n(x)$, $k = 0, \dots, n-1$.

156. (a) n; (b) $n(n+1)/2$.

157. 2.

162. yes, no, no, yes, no, yes, yes, no.

164. $E = \mathbf{va}$, i.e. $e_{ij} = v_i a_j, \ i = 1, \dots, m, \ j = 1, \dots, n$.

165. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

167. Check your answers using associativity, e.g. $(CB)A = C(BA)$.

`Other answers:` $BAC = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}, \quad ACB = \begin{bmatrix} 0 & -2 \\ 3 & 6 \end{bmatrix}$.

173. $\begin{bmatrix} \cos 1^\circ & -\sin 1^\circ \\ \sin 1^\circ & \cos 1^\circ \end{bmatrix}$.

174. If $B = A^k$, then $b_{i,i+k} = 1$, and all other $b_{ij} = 0$.

175. The answer is the same as in the previous problem.

176. (a) If A is $m \times n$, B must be $n \times m$. (b) Both must be $n \times n$.

178. Exactly when $AB = BA$.

185. Restriction of linear functions on $\mathcal{W}$ to the subspace $\mathcal{V}$.

187. I.

189. $S = 2x_1y_1 + x_1y_2 + x_2y_1$, $A = x_1y_2 - x_2y_1$.

190. No, only if $AB = BA$.

193. No; yes (of $x, y \in \mathbb{R}^1$); yes (in $\mathbb{R}^2$).

194. $(\sum x_i)(\sum y_i)$ and $\sum_{i\neq j} x_i y_j/2$.

196. $T^\dagger = \overline{z_2}\, w_1$; $S_1 = (\overline{z_1}\, w_2 + \overline{z_2}\, w_1)/2$, $S_2 = (\overline{z_1}\, w_2 - \overline{z_2}\, w_1)/2i$; $H_1 = (\overline{z_1}\, z_2 + \overline{z_2}\, z_1)/2$, $H_2 = (\overline{z_1}\, z_2 - \overline{z_2}\, z_1)/2i$.

197. $(\overline{z_1}\, w_2 + \cdots + \overline{z_{n-1}}\, w_n + \overline{z_2}\, w_1 + \cdots + \overline{z_n}\, z_{n-1})/2$.

198. $T \mapsto C^\dagger T C$.

200. No, yes, n^2.

203. 1; 1; $\mathbf{cos}(\mathbf{x} + y)$.

204. 2, 1; -2.

206. $k(k-1)/2$.

208. $n(n-1)/2 - l$.

212. Transpositions $\tau^{(i)}$ of nearby indices $(i, i+1)$, $i = 1, \ldots, n-1$.

214. E.g. $\tau_{14}\tau_{34}\tau_{25}$.

215. No; yes.

216. + (6 inverted pairs); + (8 inverted pairs).

217. $-1\,522\,200$; $-29\,400\,000$.

219. 0.

220. Changes sign, if $n = 4k + 2$ or $n = 4k + 3$ for some k, and remains unchanged otherwise.

221. $x = a_1, \ldots, a_n$.

223. Leave it unchanged.

227. (a) $(-1)^{n(n-1)/2} a_1 a_2 \cdots a_n$, (b) $(ad - bc)(eh - fg)$.

228. (a) 9, (b) 5.

229. (a) $\begin{bmatrix} -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \\ \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

230. (a) $\mathbf{x} = [-\frac{2}{9}, \frac{1}{3}, \frac{1}{9}]^t$, (b) $\mathbf{x} = [1, -1, 1]^t$.

232. $(\det A)^{n-1}$, where n is the size of A.

233. Those with determinants ± 1.

234. (a) $x_1 = 3, x_2 = x_3 = 1$, (b) $x_1 = 3, x_2 = 4, x_3 = 5$.

235. (a) $-x_1^2 - \cdots - x_n^2$, (b) $(ad - bc)^3$.

239. $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$.

240. 1.

242. $1!3!5! \cdots (2n-1)!$.

247. Use the property to establish a canonical isomorphism between two candidates on the role of $\mathcal{V} \otimes \mathcal{W}$.

262. $2(\mathbf{f}_{12} f_{34} - f_{13} f_{24} + f_{14} f_{23})$.

271. 1.

273. The system is consistent whenever $b_1 + b_2 + b_3 = 0$.

279. (a) Yes, (b) yes, (c) yes, (d) no, (e) yes, (f) no, (g) no, (h) yes.

283. $0 \le \operatorname{codim} \le k$.

285. Two subspaces are equivalent if and only if they have the same dimension.

286. The equivalence class of an ordered pair $\mathcal{U}, \mathcal{V}$ of subspaces is determined by $k := \dim U$, $l := \dim \mathcal{V}$, and $r := \dim(\mathcal{U}+\mathcal{V})$, where $k, l \le r \le n$ can be arbitrary.

287. (a) q^n; (b) $(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})$;
(c) $(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{r-1})$;
(d) $(q^n-1)(q^n-q)\cdots(q^n-q^{r-1})/(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})$.

288. $x_1 = 3, x_2 = 1, x_3 = 1$; inconsistent; $x_1 = 1, x_2 = 2, x_3 = -2$;
$x_1 = 2t_1 - t_2, x_2 = t_1, x_3 = t_2, x_4 = 1$;
$x_1 = -8, x_2 = 3 + t, x_3 = 6 + 2t, x_4 = t$; $x_1 = x_2 = x_3 = x_4 = 0$; $x_1 = \frac{3}{17}t_1 - \frac{13}{17}t_2, x_2 = \frac{19}{17}t_1 - \frac{20}{17}t_2, x_3 = t_1, x_4 = t_2$;
$x_1 = -16 + t_1 + t_2 + 5t_3, x_2 = 23 - 2t_1 - 2t_2 - 6t_3, x_3 = t_1, x_4 = t_2, x_5 = t_3$.

289. $\lambda = 5$.

290. (a) $\operatorname{rk} = 2$, (b) $\operatorname{rk} = 2$, (c) $\operatorname{rk} = 3$, (d) $\operatorname{rk} = 4$, (e) $\operatorname{rk} = 2$.

291. Inverse matrices are:

$$\begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{bmatrix}.$$

300. $\binom{n}{2} - l(\sigma)$, where $l(\sigma)$ is the length of the permutation.

301. $[n]_q! := [1]_q[2]_q \cdots [n]_q$ (called **q-factorial**), where $[k]_q := \frac{q^n-1}{q-1}$.

303. Inertia indices $(p, q) = (1, 1), (2, 0), (1, 2)$.

308. Empty for $p = 0$, has 2 components for $p = 1$, and 1 for $p = 2, 3, 4$.

312. $z_1^2 + z_2^2 = 1, z_1^2 + z_2^2 = 0, z_2 = z_1^2, z_1^2 = 1, z_1^2 = 0$.

313. Two parallel lines.

318. Those all of whose entries are imaginary.

319. $T(\mathbf{z}, \mathbf{w}) =$
$\frac{1}{2}\left[T(\mathbf{z}+\mathbf{w}, \mathbf{z}+\mathbf{w}) + iT(i\mathbf{z}+\mathbf{w}, i\mathbf{z}+\mathbf{w}) - (1+i)\left(T(\mathbf{z}, \mathbf{z}) + T(\mathbf{w}, \mathbf{w})\right)\right]$.

326. $d_{ii} = \Delta_i/\Delta_{i-1}$.

327. $(p, q) = (2, 2), (3, 1)$.

345. No.

346. The sum of the squares of all sides of a parallelogram is equal to the sum of the squares of the diagonals.

359. id.

384. $\theta(\mathbf{e}_i, \mathbf{e}_{i+1}) = 2\pi/3$, while all other pairs are perpendicular.

386. One can take $\mathbf{f}_i = \mathbf{e}_i - \mathbf{e}_0$, and $\mathbf{h}_i = \mathbf{e}_i - \mathbf{e}_{i-1}$, where $i = 1, \ldots n$.

388. Rotations about the origin, and reflections about a line passing thorough the origin, which have determinants 1 and -1 respectively.

394. ± 1.

401. The operator is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

427. $(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $(b) \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(c) \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
$(e) \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $(f) \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $(k) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(l) \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$.

437. $(a_0 - n(a_0 - a_1/3))3^n$.

438. (a) $(1 + \sqrt{10})/3$.

441. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ will do.

448. Pairs of $m \times n$-matrices; pairs of matrices of transposed sizes.

452. $\mathbb{K}^1$.

454. The restriction to the eigenline.

459. (a) $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$; (b) $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$.

461. Symmetry planes of regular polyhedron (tetrahedron in case (b), octahedron or cube in case (c), and icosahedron or dodecahedron in case (d)) partition the surface of the sphere circumscribed around the polyhedron into (respectively, 24, 48, and 120) required spherical triangles.

462. Since $\langle \mathbf{v}, \mathbf{v} \rangle = 2$, we have:
$\langle \mathbf{x} - \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v},\ \mathbf{y} - \langle \mathbf{y}, \mathbf{v} \rangle \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{v} \rangle \langle \mathbf{x}, \mathbf{v} \rangle + 2 \langle \mathbf{x}, \mathbf{v} \rangle \langle \mathbf{y}, \mathbf{v} \rangle$
$= \langle \mathbf{x}, \mathbf{y} \rangle$.

464. $\mathcal{V}^{i,j} : \cdots \to 0 \to \mathbb{K}^1 \stackrel{\cong}{\to} \cdots \stackrel{\cong}{\to} \mathbb{K}^1 \to 0 \to \ldots$ (i zeroes followed by $j - i$ copies of $\mathbb{K}^1$, followed by $n - j$ zeroes).

465. $\mathcal{V}^{0,n} \oplus \mathcal{V}^{1,n} \oplus \cdots \oplus \mathcal{V}^{n-1,n}$.

466. $X(t) = e^{tA} X(0) e^{-tA}$.

467. Eigenvalues are $\lambda_i - \lambda_j$, $i, j = 1, \ldots, n$. In the space of linear forms, they form the root system of type A_{n-1}. Eigenvectors are matrices E_{ij} whose all entries are 0 except a single 1 in the ith row and jth column.

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