Hints

8. \(\overrightarrow{AA'} = 3\overrightarrow{AM}/2\).
14. \(\cos^2 \theta \leq 1\).
19. Rotate the regular \(n\)-gon through \(2\pi/n\).
22. Take \(X = A\) first.
23. \(\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}\).
24. If \(a + b + c + d = 0\), then \(2(b + c)(c + d) = a^2 - b^2 + c^2 - d^2\).
25. \(XA^2 = |\overrightarrow{OX} - \overrightarrow{OA}|^2 = 2R^2 - 2(\overrightarrow{OX}, \overrightarrow{OA})\) if \(O\) is the center.
26. Consider projections of the vertices to any line.
27. Project faces to an arbitrary plane and compute signed areas.
35. Compute \((\cos \theta + i \sin \theta)^3\).
37. Divide \(P\) by \(z - z_0\) with a remainder.
40. \((z - z_0)(z - \overline{z}_0)\) is real.
41. Apply Vieta’s formula.
45. \(e^{\pm i\theta} = \cos \theta \pm i \sin \theta\).
46. See Figure 13.
50. Show first that \(\binom{n}{k}\) equals the number of \(k\)-element subsets in the set of \(n\) objects.
52. Show that the sum of distances to the foci of an ellipse from points on a tangent line is minimal at the tangency point; then reflect the source of light about the line.
54. \(e = |P'P|/|P'Q|\) is the slope of the sectioning plane.
55. Apply the forthcoming classification of quadratic curves.
56. \(AX^2 + CY^2\) is even in \(X\) and \(Y\).
57. \(Q(-x, y) = Q(x, y)\) only if \(b = 0\).
60. \(\cos \theta = 4/5\).
62. Rotating the coordinate system as in Example on p. 11, don’t forget to apply the change to the linear terms.
63. $Q(-x, -y) = Q(x, y)$ for a quadratic form $Q$.

65. Reflect the plane about a suitable line.

68. $x^2 + 4xy = (x + 2y)^2 - 4y^2$.

69. Examine when the quadratic equation $at^2 + 2bt + c = 0$ has a double root.

76. Given that $A$ and $B$ can be transformed into $C$, find a way to transform $A$ into $B$.

80. For each of the normal forms, draw the region of the plane where the quadratic form is positive.

81. Draw the region on the plane where $xy > 0$.

85. First apply the Inertia Theorem to make $Q = x_1^2 + x_2^2$.

91. Take $x_1 = x$, $x_2 = \dot{x}$.

97. In the “toy version,” one of the quadratic forms is $x^2 + y^2$, the squared Euclidean distance to the origin.

98. $4ac = (a + c)^2 - (a - c)^2$.

102. $\frac{3n(n+1)}{2} - n^2 = \frac{n(n+3)}{2} > \frac{n^2}{2}$.

103. $1 \neq 0$.

105. If $ba = 1$ and $ab' = 1$, then $b = bab' = b'$.

110. This is a rephrasing of the previous exercise.

114. If $(a, n) = 1$, then $ka + ln = 1$ for some integers $k$ and $l$.

116. Show that a non-zero homomorphism between two fields maps $1$ to $1$, and inverses to inverses.

118. Construct the homomorphism $f : \mathbb{Z} \to \mathbb{K}$ such that $f(1) = 1$.

119. Take $\mathbb{F} = \{0, 1, \eta, \zeta\}$, and set $1 + 1 = 0$, $\eta + \zeta = 1$, $\eta \zeta = 1$.

120. $x^2 + x + 1 = (x + \eta)(x + \zeta)$.

121. Compute $0 + 0'$.

122. $-u = (-1)u$.

123. Redefine multiplication by scalars as $\lambda u = 0$ for all $\lambda \in \mathbb{K}$ and all $u \in \mathcal{V}$.

124. $0u = (0 + 0)u = 0u + 0u$.

129. Rotated parallelograms are still parallelograms.

132. A linear function on $\mathcal{V} \oplus \mathcal{W}$ is uniquely determined by its restrictions to $\mathcal{V}$ and $\mathcal{W}$.

135. Translate given linear subspaces by any vector in the intersection of the affine ones.

141. Examine the kernel of the map $\mathbb{R}[x] \to \mathbb{R}^2$ which associates to a polynomial $P$ the pair $(P(1), P(-1))$ of its values at $x = \pm 1$.

152. $\cos 4\theta + i\sin 4\theta = (\cos \theta + i\sin \theta)^4$. 
Hints

154. Prove that $E$ is injective by identifying $\mathcal{V}$ with $\mathbb{K}^n$ and evaluating coordinate functions on a non-zero vector.

158. To a linear form $f : \mathcal{V}/\mathcal{W} \to \mathbb{K}$, associate $\mathcal{V} \overset{\pi}{\to} \mathcal{V}/\mathcal{W} \overset{f}{\to} \mathbb{K}$.

160. $|\mathbb{F}| = p^n$, where $n$ is the dimension of $\mathbb{F}$ as a $\mathbb{Z}_p$-vector space.

167. Both associate to an input $x \in X$ the output $w = f(g(h(x))$.

176. Pick two $2 \times 2$-matrices at random.

179. Compute $BAC$, where $B$ and $C$ are two inverses of $A$.

180. Given two isomorphisms $B : \mathcal{U} \to \mathcal{V}$ and $A : \mathcal{V} \to \mathcal{W}$, find an isomorphism from $\mathcal{W}$ to $\mathcal{U}$.

181. Consider $1 \times 2$ and $2 \times 1$ matrices.

183. Solve $D^{-1}AC = I$ for $D$ and $C$.

187. Which $2 \times 2$-matrices are anti-symmetric?

191. Start with any non-square matrix.

201. Each elementary product contains a zero factor.

204. There are 12 even and 12 odd permutations.

206. Permutation $\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$ has length $\binom{n}{2}$.

208. Pairs of indices inverted by $\sigma$ and $\sigma^{-1}$ are the same.

209. The transformation: $\text{logarithm} \to \text{algorithm} \to \text{algorithm}$ consists of two transpositions.

212. Locate a pair $(i, i+1)$ of nearby indices which $\sigma$ inverses, compose $\sigma$ with the transposition of these indices, and show that the length decreases by 1.

217. $247 = 2 \cdot 100 + 4 \cdot 10 + 7$.

221. $\det(-A) = (-1)^n \det A$.

223. $\det(C^tQC) = (\det Q)(\det C)^2$.

234. Apply the 1st of the “cool formulas.”

235. Compute the determinant of a $4 \times 4$-matrix the last two of whose rows repeat the first two.

236. Apply Binet–Cauchy’s formula to the $2 \times 3$ matrix whose rows are $x^t$ and $y^t$.

237. Show that $\Delta_n = a_n\Delta_{n-1} + \Delta_{n-2}$.

239. Use the defining property of Pascal’s triangle: $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$.

240. Use the fact of algebra that a polynomial in $(x_1, \ldots, x_n)$, which vanishes when $x_i = x_j$, is divisible by $x_i - x_j$.

241. Divide the $k$th column by $k$ and apply Vandermonde’s identity.

242. Product $p(x)q(y)$ of polynomials is linear in $p$ and $q$. 
243. $p(s)q(s')$.

245. $f \otimes w$ corresponds to the rank-1 linear map $v \mapsto f(v)w$.

248. Use universality.

251. $H : \mathbb{R}^7 \to \mathbb{R}^1$.

252. The space spanned by columns of $A$ and $B$ contains columns of $A + B$.

260. $(A^t a)(x) = a(Ax)$ for all $x \in \mathbb{K}^n$ and $a \in (\mathbb{K}^m)^*$.

274. Modify the Gaussian elimination algorithm of Section 2 by permuting unknowns (instead of equations).

275. Apply the $LUP$ decomposition to $M^t$.

276. Apply $LPU$, $LUP$, and $PLU$ decompositions to $M^{-1}$.

281. When $\mathbb{K}$ has $q$ elements, each cell of dimension $l$ has $q^l$ elements.

285. Consider the map $\mathbb{R}^n \to \mathbb{R}^{p+q}$ defined by the linear forms.

298. $T = A + iB$ where both $A$ and $B$ are Hermitian.

299. $\langle x, y \rangle = z^\dagger w$.

301. $\langle z, w \rangle = z^\dagger w$.

302. Use the previous exercise.

304. The normal forms are: $i|Z_1|^2 - i|Z_2|^2$, $|Z_1|^2 - |Z_2|^2$, $|Z_1|^2$.

309. Take the linear form for one of new coordinates.

311. Prove that $-1$ is a non-square.

314. There are 1 even and 3 odd non-degenerate forms.

318. $\frac{1}{3} = 1 - \frac{2}{3}$.

319. Start with inverting 2-adic units $\cdots \ast \ast \ast 1$. ($\ast$ is a wild card).

321. Use the previous exercise.

345. Apply Vieta’s formula.

347. Compute $\langle x, Ax \rangle$ for $x = \sum t_i v_i$, where $\{v_i\}$ is an orthonormal basis of eigenvectors, and $\sum t_i^2 = 1$.

348. If $AB = BA$, then eigenspaces of $A$ are $B$-invariant.

353. Find an eigenvector, and show that its orthogonal complement is invariant.

357. Construct $v_1, \ldots, v_n$ by applying the Spectral Theorem to the non-negative Hermitian operator $A^\dagger A$, and check that $Av_i \in \mathcal{W}$ corresponding to distinct nonzero eigenvalues $\mu_i$ of $A^\dagger A$ are pairwise perpendicular.

358. Rephrase the previous exercise in matrix form.

361. The equation $\langle x + ty, x + ty \rangle = 0$ quadratic in $t$ has no real solutions unless $x$ is proportional to $y$.

363. $\langle x, x \rangle = (\sum x_i)^2 + \sum x_i^2$.

365. $\det(U^t U) = 1$. 
See Example 1.

Use the previous exercise.

Prove first that the radius of the circle must be equal to the middle semiaxis of the ellipsoid.

For “exactly one” claim, use the coordinate-less description of the spectrum provided by the minimax principle.

When \( x \approx 0 \), \( \sin x \approx x \).

If \( T \) denotes the cyclic shift of coordinates in \( \mathbb{C}^n \), then \( C = C_0I + C_1T + \cdots + C_{n-1}T^{n-1} \).

Use Newton’s binomial formula.

Pick \( A \) and \( B \) so that \( e^A e^B \neq e^B e^A \).

Verify it first for a Jordan cell.

Make two of the lines the coordinate axes on the plane, and the third one the graph of an isomorphism between them.

If the first three lines are \( y = 0, x = 0 \), and \( y = x \), then the fourth one is \( y = \lambda x \).

If \( d = 0, a = \infty, b = 1 \), then \( \lambda = c \).

Assuming that the spaces \( \mathcal{U}_i \) have the same dimension, and the maps between them are invertible, consider the determinant (or eigenvalues) of the composition of the maps around the cycle.

Use the theory of Bruhat cells.

Use Lemma, and apply induction on the length of the shortest of the legs.

Use Sylvester’s rule together with the previous exercise.

On the unit sphere of area \( 4\pi \), the area of a triangle with the angles \( (\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}) \) has the area \( \frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} – \pi \). Use this to compute the number of required triangles.

Compute \( R_{\mathbf{v}_1} R_{\mathbf{v}_2} \) to show that it is a Jordan cell of size 2 with the eigenvalue 1.

They correspond to positive roots \( \mathbf{e}_i - \mathbf{e}_j = \mathbf{v}_{i+1} + \cdots + \mathbf{v}_j \), where \( 0 \leq i < j \leq n \).
Answers

1. $mg/2$, $mg\sqrt{3}/2$.
2. 18 min (reloading time excluded).
6. It rotates with the same angular velocity along a circle centered at the barycenter of the triangle formed by the centers of the given circles.
10. $3/4$.
13. $7\overrightarrow{OA} = \overrightarrow{OA} + 2\overrightarrow{OB} + 4\overrightarrow{OC}$ for any $O$.
15. $3/2$.
17. $2\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u} + \mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2$.
18. (b) No.
28. Yes; Yes (0); Yes.
29. (a) $1 + 5i/3$; (b) $1 - i\sqrt{3}/2$.
31. $|z|^{-2}$.
33. (a) $\sqrt{2}, -\pi/4$; (b) $2, -\pi/3$.
34. $-\frac{1 + i\sqrt{3}}{2}$.
38. $2 \pm i; \frac{1 \pm \sqrt{5}}{2}; 1 + i, 1 + i; 1 \pm \frac{\sqrt{5} - i\sqrt{2}}{2}$.
39. $-2, 1 \pm i\sqrt{3}; i, \frac{\pm \sqrt{3} - i}{2}; \pm i\sqrt{2}, \pm i\sqrt{2}; \frac{\sqrt{3} + i}{2}, \frac{\sqrt{3} - i}{2}; \pm i, \pm \frac{\sqrt{3} + i}{2}$.
42. $a_k = (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} z_{i_1} \cdots z_{i_k}$.
45. $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin \theta = (e^{i\theta} - e^{-i\theta})/2i$.
47. The upper half-plane $\text{Im } w > 0$.
49. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.
58. Yes, $k(x^2 + y^2)$.
59. $y = \pm x$; level curves of (a) are ellipses, of (c) hyperbolas.
60. $2X^2 - Y^2 = 1$.
62. Semiaxes $2/\sqrt{3}$ and 2.
64. (a) \( \pm (\sqrt{\alpha^2 - \beta^2}, 0) \), (b) \( \pm (\sqrt{\alpha^2 + \beta^2}, 0) \).

68. 2nd ellipse, 1st & 4th hyperbolas.

72. A pair of intersecting lines, \( x - 1 = \pm (2y - 1) \).

79. For normal forms, \( X^2 \) and 0 can be taken.

84. \( \frac{(n+1)(n+2)}{2} \).

88. \( x(t) = x(0)e^{\lambda t} \).

89. \( x(t) = x(0)e^{3t}, y(t) = y(0)e^{-t}, z(t) = z(0) \).

92. \( \dot{X}_1 = iX_1, \dot{X}_2 = -iX_2 \).

96. Yes, some non-equivalent equations represent the empty set.

101. \( \frac{n(n+1)}{2} \).

107. Use the **Euclidean algorithm**: divide \( a \) by \( b \) with the remainder \( r \), then divide \( b \) by \( r \) with the remainder \( r_1 \), then divide \( r \) by \( r_1 \) with the remainder \( r_2 \), etc. until the remainder becomes 0.

108. Use the previous exercises.

111. \( n \).

113. 1, 3, 5, 7.

134. Points, lines, and planes.

137. \( p^n; p^{mn} \).

138. \( p^{n(n-1)/2} \).

140. (b) \( \text{Ker } D = \mathbb{K}[x^p] \subset \mathbb{K}[x] \).

149. \( \text{dim } = 2 \).

151. \( x^k = x_1^kL_1(x) + \cdots + x_n^kL_n(x), \ k = 0, \ldots, n-1 \).

155. (a) \( n \); (b) \( n(n+1)/2 \).

156. 2, if \( n > 1 \).

161. yes, no, no, yes, no, yes, yes, no.

163. \( E = va \), i.e. \( e_{ij} = v_ia_j, \ i = 1, \ldots, m, \ j = 1, \ldots, n \).

164. \( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \).

166. Check your answers using associativity, e.g. \( (CB)A = C(BA) \).

\[ \text{Other answers: } BAC = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}, \quad ACB = \begin{bmatrix} 0 & -2 \\ 3 & 6 \end{bmatrix}. \]

172. \( \begin{bmatrix} \cos 1^\circ & -\sin 1^\circ \\ \sin 1^\circ & \cos 1^\circ \end{bmatrix} \).

173. If \( B = A^k \), then \( b_{i,i+k} = 1 \), and all other \( b_{ij} = 0 \).

174. The answer is the same as in the previous problem.
175. (a) If $A$ is $m \times n$, $B$ must be $n \times m$. (b) Both must be $n \times n$.
177. Exactly when $AB = BA$.
184. Restriction of linear functions on $\mathcal{W}$ to the subspace $\mathcal{V}$.
186. I.
188. $S = 2x_1y_1 + x_1y_2 + x_2y_1$, $A = x_1y_2 - x_2y_1$.
189. No, only if $AB = BA$.
192. No; yes (of $x, y \in \mathbb{R}^1$); yes (in $\mathbb{R}^2$).
193. $(\sum x_i)(\sum y_i)$ and $\sum_{i \neq j} x_iy_j / 2$.
195. $T^\dagger = \bar{w}_2 w_1$; $S_1 = (\bar{w}_1 w_2 + \bar{w}_2 w_1) / 2$, $S_2 = (\bar{w}_1 w_2 - \bar{w}_2 w_1) / 2i$; $H_1 = (\bar{z}_1 z_2 + \bar{z}_2 z_1) / 2$, $H_2 = (\bar{z}_1 z_2 - \bar{z}_2 z_1) / 2i$.
196. $(\sum x_i w_i + \cdots + \sum_{n-1} w_i + \bar{w}_2 w_1 + \cdots + \bar{w}_n w_{n-1}) / 2$.
197. $T \rightarrow C^\dagger TD$.
199. No, yes, $n^2$.
202. 1; 1; $\cos(x + y)$.
203. 2, 1; $-2$.
205. $k(k - 1)/2$.
207. $n(n - 1)/2 - l$.
211. Transpositions $\tau^{(i)}$ of nearby indices $(i, i + 1)$, $i = 1, \ldots, n - 1$.
213. E.g. $\tau_{14}\tau_{34}\tau_{25}$.
214. No; yes.
215. + (6 inverted pairs); + (8 inverted pairs).
216. $-1522200; -29400000$.
218. 0.
219. Changes sign, if $n = 4k + 2$ or $n = 4k + 3$ for some $k$, and remains unchanged otherwise.
220. $x = a_1, \ldots, a_n$.
222. Leave it unchanged.
226. (a) $(-1)^{n(n-1)/2}a_1 a_2 \cdots a_n$, (b) $(ad - bc)(eh - fg)$.
227. (a) 9, (b) 5.
228. (a) \[
\begin{bmatrix}
-\frac{5}{18} & -\frac{7}{18} & \frac{1}{18} \\
\frac{1}{18} & \frac{2}{18} & -\frac{1}{18} \\
\frac{1}{18} & \frac{1}{18} & -\frac{5}{18}
\end{bmatrix},
\] (b) \[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}.
\]
229. (a) $x = [-\frac{2}{9}, \frac{1}{3}, \frac{1}{9}]^t$, (b) $x = [1, -1, 1]^t$.
231. $(\det A)^{n-1}$, where $n$ is the size of $A$.
232. Those with determinants $\pm 1$.
233. (a) $x_1 = 3, x_2 = x_3 = 1$, (b) $x_1 = 3, x_2 = 4, x_3 = 5$.
234. (a) $-x_1^2 - \cdots - x_n^2$, (b) $(ad - bc)^3$. 
238. $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$.

239. 1.

241. $1!3!5! \cdots (2n - 1)!$.

246. Use the property to establish a canonical isomorphism between two candidates on the role of $\mathcal{V} \otimes \mathcal{W}$.

251. 1.

253. The system is consistent whenever $b_1 + b_2 + b_3 = 0$.

259. (a) Yes, (b) yes, (c) yes, (d) no, (e) yes, (f) no, (g) no, (h) yes.

263. $0 \leq \text{codim} \leq k$.

265. Two subspaces are equivalent if and only if they have the same dimension.

266. The equivalence class of an ordered pair $\mathcal{U}, \mathcal{V}$ of subspaces is determined by $k := \dim \mathcal{U}, l := \dim \mathcal{V}$, and $r := \dim(\mathcal{U} + \mathcal{V})$, where $k, l \leq r \leq n$ can be arbitrary.

267. (a) $q^n$; (b) $(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$; (c) $(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$; (d) $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})/(q^r - 1)(q^r - q) \cdots (q^r - q^{r-1})$.

268. $x_1 = 3, x_2 = 1, x_3 = 1$; inconsistent; $x_1 = 1, x_2 = 2, x_3 = -2$; $x_1 = 2t_1 - t_2, x_2 = t_1, x_3 = t_2, x_4 = 1$; $x_1 = -8, x_2 = 3 + t, x_3 = 6 + 2t, x_4 = t$; $x_1 = x_2 = x_3 = x_4 = 0$;

269. $\lambda = 5$.

270. (a) $\text{rk} = 2$, (b) $\text{rk} = 2$, (c) $\text{rk} = 3$, (d) $\text{rk} = 4$, (e) $\text{rk} = 2$.

271. Inverse matrices are:

$$
\begin{bmatrix}
1 & -4 & -3 \\
1 & -5 & -3 \\
-1 & 6 & 4
\end{bmatrix}, \quad \frac{1}{4}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix},
\begin{bmatrix}
2 & -1 & 0 & 0 \\
3 & 2 & 0 & 0 \\
31 & -19 & 3 & -4 \\
-23 & 14 & -2 & 3
\end{bmatrix}.
$$

279. $\binom{n}{2} - l(\sigma)$, where $l(\sigma)$ is the length of the permutation.

280. $[n]_q! := [1]_q[2]_q \cdots [n]_q$ (called $q$-factorial), where $[k]_q := \frac{q^k - 1}{q - 1}$.

282. Inertia indices $(p, q) = (1, 1), (2, 0), (1, 2)$.

287. Empty for $p = 0$, has 2 components for $p = 1$, and 1 for $p = 2, 3, 4$.

291. $z_1^2 + z_2^2 = 1, z_1^2 + z_2^2 = 0, z_2 = z_1^2, z_1^2 = 1, z_1^2 = 0$.

292. Two parallel lines.

297. Those all of whose entries are imaginary.

298. $T(z, w) = \frac{1}{2} [T(z + w, z + w) + iT(i(z + w, i(z + w)) - (1 + i)(T(z, z) + T(w, w))].$
305. \( d_{ii} = \Delta_i/\Delta_{i-1} \).

306. \((p, q) = (2, 2), (3, 1)\).

324. No.

325. The sum of the squares of all sides of a parallelogram is equal to the sum of the squares of the diagonals.

338. id.

362. \( \theta(e_i, e_{i+1}) = 2\pi/3 \), while all other pairs are perpendicular.

364. One can take \( f_i = e_i - e_0 \), and \( h_i = e_i - e_{i-1} \), where \( i = 1, \ldots n \).

366. Rotations about the origin, and reflections about a line passing through the origin, which have determinants 1 and \(-1\) respectively.

379. The operator is \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

405. (a) \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \), (b) \( \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), (c) \( \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \).

415. \( (a_0 - n(a_0 - a_1/3))^3 \).

416. (a) \((1 + \sqrt{10})/3\).

419. \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) will do.

426. Pairs of \( m \times n \)-matrices; pairs of matrices of transposed sizes.

430. \( \mathbb{K}^1 \).

432. The restriction to the eigenline.

437. (a) \((2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)\); (b) \((3, 3, 3), (2, 4, 4), (2, 3, 6)\).

439. Symmetry planes of regular polyhedron (tetrahedron in case (b), octahedron or cube in case (c), and icosahedron or dodecahedron in case (d)) partition the surface of the sphere circumscribed around the polyhedron into (respectively, 24, 48, and 120) required spherical triangles.

440. Since \( \langle v, v \rangle = 2 \), we have:
\[
\langle x - (x, v)v, y - (y, v)v \rangle = \langle x, y \rangle - \langle x, v \rangle \langle v, y \rangle - \langle y, v \rangle \langle x, v \rangle + 2 \langle x, v \rangle \langle y, v \rangle = \langle x, y \rangle.
\]

442. \( \mathcal{V}^{i,j} : \cdots \rightarrow 0 \rightarrow \mathbb{K}^1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathbb{K}^1 \rightarrow 0 \rightarrow \cdots \) (\( i \) zeroes followed by \( j - i \) copies of \( \mathbb{K}^1 \), followed by \( n - j \) zeroes).

443. \( \mathcal{V}^{0,n} \oplus \mathcal{V}^{1,n} \oplus \cdots \oplus \mathcal{V}^{n-1,n} \).

444. \( X(t) = e^{tA}X(0)e^{-tA} \).
445. Eigenvalues are $\lambda_i - \lambda_j$, $i, j = 1, \ldots, n$. In the space of linear forms, they form the root system of type $A_{n-1}$. Eigenvectors are matrices $E_{ij}$ whose all entries are 0 except a single 1 in the $i$th row and $j$th column.
Index

n-space, 43
n-vectors, 43

absolute value, 12
addition of vectors, 39
additivity, 7
adjoint form, 70
adjoint map, 65, 102
adjoint systems, 105
adjugate matrix, 84
affine subspace, 45, 49
algebraic number, 40
algebraically closed field, 174
annihilator, 57
anti-Hermitian form, 71, 127
anti-Hermitian quadratic form, 127
anti-linear, 70
anti-symmetric bilinear form, 68
anti-symmetric matrix, 68
anti-symmetric operator, 155
argument of complex number, 13
associative, 62
associativity, 4
augmented matrix, 108
axiom, 39
axis of symmetry, 25

Bézout, 17
Bézout’s theorem, 17
back substitution, 108
barycenter, 4
basis, 5, 51
bijective, 44
bilinear form, 67
bilinearity, 8
Binet, 89
Binet–Cauchy formula, 89
block, 79
block triangular matrix, 79
Bruhat cell, 119
canonical form, 29
canonical isomorphism, 56
canonical projection, 49
Cartesian coordinates, 9
category of vector spaces, 44
Cauchy, 89
Cauchy – Schwarz inequality, 9, 139
Cauchy’s interlacing theorem, 160
Cayley transform, 143
characteristic, 42
characteristic equation, 144
characteristic polynomial, 35, 167
Chebyshev, 55
Chebyshev polynomials, 55
classification theorem, 29
codimension, 104
cofactor, 83
cofactor expansion, 83, 87
column, 47
column space, 111
commutative square, 101, 193
commutativity, 4
commutator, 129
complementary multi-index, 87
complete flag, 117
completing squares, 26
complex conjugate, 11
complex conjugation, 153
complex multiplication, 157
complex sphere, 125
complex vector space, 40
complexification, 153
components, 47
components of tensor, 96
composition, 61
Index

congruent modulo $n$, 41
conic section, 21
conics, 124
connected graph, 191
continuous time, 179
contravariant, 96
coordinate Euclidean space, 151
coordinate flag, 118
coordinate system, 5
coordinate vectors, 43, 47
coordinates, 5, 43, 47, 53
covariant, 96
covector, 95
Cramer’s rule, 85
cross product, 90
cross-ratio, 197
cylinder, 124

Dandelin, 22
Dandelin’s spheres, 21
Darboux, 135
Darboux basis, 135
decomposable representation, 194
degenerate bilinear form, 159
Descartes, 9
determinant, 37, 73
diagonal matrix, 47
diagonalizable matrix, 173
dimension, 52
dimension of Bruhat cell, 119
direct sum, 32, 44, 194
direct sum of representations, 194
directed segment, 3
directrix, 23
discrete dynamical systems, 179
discrete time, 179
discriminant, 14, 138
distance, 139
distributive law, 12, 60
distributivity, 4
dot product, 7
dot-product, 67
dual basis, 54, 95
dual map, 65
dual space, 46
eccentricity, 23
edge, 191
eigenspace, 145, 167
eigenvalue, 33, 144, 167
eigenvector, 144, 167
Einstein convention, 95
elementary product, 73
elementary row operations, 107
ellipse, 21
ellipsoid, 161
entry, 47
equivalent, 29
equivalent conics, 124
equivalent linear maps, 101
equivalent representations, 192
Euclidean algorithm, 210
Euclidean inner product, 151
Euclidean space, 151
Euclidean structure, 151
Euler’s formula, 18
evaluation, 60
evaluation map, 48
even form, 134
even permutation, 75
exponential function, 17
Fibonacci, 179
Fibonacci sequence, 179
field, 12, 40
field of $p$-adic numbers, 137
finite dimensional spaces, 52
flag, 117
focus of ellipse, 22
focus of hyperbola, 23
focus of parabola, 23
Fourier, 164
Fourier basis, 164
Fundamental Formula of Mathematics, 19
Gabriel, 195
Gaussian elimination, 107
Gelfand’s problem, 159
golden ratio, 180
graded algebra, 96
Gram matrix, 141
Gram–Schmidt process, 131, 140
graph, 45, 191
gravitation constant, 166
greatest common divisor, 41
group, 202
half-linear, 70
Hamilton–Cayley equation, 175
harmonic oscillator, 163
Hasse, 137
head, 3
Hermite, 70
Hermite polynomials, 55
Hermitian adjoint, 142
Hermitian adjoint form, 70
Hermitian conjugate matrix, 70
Hermitian form, 71
Hermitian inner product, 139
Hermitian isomorphic, 140
Hermitian operator, 142
Hermitian quadratic form, 71, 126
Hermitian space, 139
Hermitian-anti-symmetric form, 70
Hermitian-symmetric form, 70
Hilbert space, 139
homogeneity, 7
homogeneous system, 104
homomorphism, 41
homomorphism theorem, 50
hyperbola, 22
hyperplane, 106
hypersurface, 124

identity matrix, 63
identity permutation, 75
imaginary part, 11
imaginary unit, 11
inconsistent system, 109
indecomposable representation, 194
indices in inversion, 75
induction hypothesis, 53
inertia index, 122
Inertia Theorem, 28
injective, 44
inner product, 7
inverse subspace, 145
inverse matrix, 63
inverse transformation, 64
inversion of indices, 75
involution, 153
irreducible representation, 195
isometric Hermitian spaces, 140

isomorphic spaces, 44
isomorphism, 44
iterations of linear maps, 179

Jordan block, 35
Jordan canonical form, 172
Jordan cell, 172
Jordan normal form, 172
Jordan system, 35

kernel, 44
kernel of form, 123, 126
kinetic energy, 162
Kronecker delta, 95

Lagrange polynomials, 55
Laplace, 87
Laplace’s formula, 87
law of cosines, 9
LDU decomposition, 116
leading coefficient, 108
leading entry, 108
leading minors, 129
left inverse, 102
length, 139
length of permutation, 75
linear combination, 4
linear dynamical systems, 179
linear form, 45, 59
linear function, 45, 59
linear map, 44
linear ODE, 179
linear recursion relation, 179
linear subspace, 43
linear transformation, 64
linearly dependent, 52
linearly independent, 52
Linnaeus, 29
lower triangular, 114
lower-triangular matrix, 47
LPU decomposition, 114
LU decomposition, 116
LUP decomposition, 117

Möbius band, 56
mathematical induction, 53
matrix, 47, 59
matrix entry, 59
matrix product, 60, 61
metric space, 139
Minkowski, 137
Minkowski–Hasse theorem, 137
minor, 83, 87
multi-index, 87
multiplication by scalar, 3
multiplication by scalars, 39
multiplicative, 12
multiplicity, 14

nilpotent operator, 169
non-degenerate bilinear form, 159
non-degenerate Hermitian form, 129
nontrivial linear combination, 52
normal form, 29
normal operator, 142, 155
null space, 44, 111

odd form, 134
odd permutation, 75
ODE, 179
operator, 142
opposite coordinate flag, 118
opposite vector, 39
oriented graph, 191
orthogonal, 140
orthogonal basis, 120
orthogonal complement, 145
orthogonal diagonalization, 149
Orthogonal Diagonalization Theorem, 159
orthogonal projection, 141
orthogonal projector, 143
orthogonal transformation, 151
orthogonal vectors, 9
orthonormal basis, 130, 140, 151

parabola, 27
partition, 171
pendulum, 166
permutation, 73
permutation matrix, 114
phase plane, 163
pivot, 108
Plücker, 90
Plücker coordinates, 104
Plücker identity, 90
PLU decomposition, 117

polar, 13
polar decomposition, 150
polylinear form, 93
positive definite, 121
positive operator, 149
positivity, 7
potential energy, 162
power of matrix, 65
principal axes, 25, 161
principal minor, 122
projection, 140
Pythagorean theorem, 9
q-factorial, 119, 212
quadratic curve, 21
quadratic form, 23, 31, 69
quadratic formula, 14
quiver, 191
quotient space, 49
range, 44
rank, 31, 99
rank of linear system, 104
rank of matrix, 100
real normal operators, 152
real part, 11
real spectral theorem, 155
real vector space, 40
realification, 153
reduced row echelon form, 108
reducible row echelon form, 108
reflection, 202
regular nilpotent, 169
representation, 191
right inverse, 102
root of unity, 15
root space, 168
root system, 203
row echelon form, 108
row echelon form of rank r, 108
row space, 111
scalar, 39, 40
scalar product, 7
semi-axes, 161
semiaxis of ellipse, 26
sesquilinear form, 70, 126
sign of permutation, 73
Index

similarity, 167
similarity transformation, 64
simple problems, 37
simple quiver, 195
singular value decomposition, 149
span, 51
spectral theorem, 144
spectrum, 33, 159
square matrix, 47, 63
square root, 15, 149
standard basis, 51
standard coordinate flag, 117
standard coordinate space, 43
standard Euclidean space, 71
standard Hermitian space, 71
subrepresentation, 195
subspace, 43
surjective, 44
Sylvester, 129
symmetric bilinear form, 68, 120
symmetric matrix, 68
symmetric operator, 155
symmetric tensors, 97
symmetricity, 7
system of linear equations, 61
tail, 3
tautology, 50
tensor, 96
tensor algebra, 96
tensor product, 93
time-independent dynamical system, 179
total anti-symmetry, 77
trace, 143
transition matrix, 63, 95
transposed form, 68
transposed map, 65
transposed matrix, 66
transposed partition, 171
transposition matrix, 114
transposition permutation, 75
triangle inequality, 9, 139
unipotent, 172
unipotent matrix, 116
unit coordinate vectors, 51
unit vector, 8
unitary rotation, 147
unitary space, 139
unitary transformation, 143
universality property, 94
upper triangular, 114
upper-triangular matrix, 47
Vandermonde, 91
Vandermonde's identity, 91
vector, 3, 39
vector space, 36, 39
vector subspace, 43
vector sum, 3
vertex, 191
Vieta, 17
Vieta's theorem, 16
Young tableaux, 171
zero representation, 194
zero vector, 3, 39