Epilogue: Quivers

Gabriel's Theorem

A figure consisting of several points connected by edges is called a **graph**. More precisely, a graph is a purely combinatorial object, which is considered given, if a finite set of its **vertices** has been specified, and the pairs of vertices connected by **edges** have been specified too. If the edges are equipped with directions, the graph is called **oriented**. A graph is called **connected** if between any two vertices there exists a path consisting of edges (regardless of their directions).

A **quiver** is a connected oriented graph. For some examples see Figure 47, which shows that we do not exclude the possibility of multiple edges connecting the same vertices, or edges connecting a vertex with itself.

Given a quiver, its **representation** consists of vector spaces assigned to the vertices, and linear maps assigned to the edges. More precisely, to each vertex v_i there should correspond a finite dimensional $\mathbb{K}$-vector space $\mathcal{V}_i$, and to each edge e_{ij} directed from v_i to v_j, there should correspond a $\mathbb{K}$-linear map $A_{ij} : \mathcal{V}_i \to \mathcal{V}_j$.

`Examples` (see Figure 47). `1.` The quiver, called A_1, consists of one vertex with no edges. A representation of this quiver is simply a vector space.

`2.` The quiver called $\tilde{A}_0$ consists of one vertex and one edge from this vertex to itself. A representation of this quiver is a linear map from a vector space to itself.

`3.` The quiver called A_2 consists of two vertices connected by an edge. A representation of this quiver is a linear map between two vector spaces.

`4.` In $\mathbb{K}^n$, consider a complete flag $\mathcal{V}^1 \subset \mathcal{V}^2 \subset \cdots \subset \mathcal{V}^{n-1} \subset \mathbb{K}^n$. It can be interpreted as a representation of the quiver consisting of n vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ (the case of $n = 5$ is shown on Figure

47 under the name A_5) connected by the edges $e_{12}, e_{23}, \dots, e_{n-1,n}$. Indeed, the spaces of the flag correspond to the vertices, and the inclusions $\mathcal{V}^i \subset \mathcal{V}^{i+1}$ are the required linear maps. Of course, not every representation of this quiver corresponds to a flag (and in particular, the linear maps are not required to be injective).

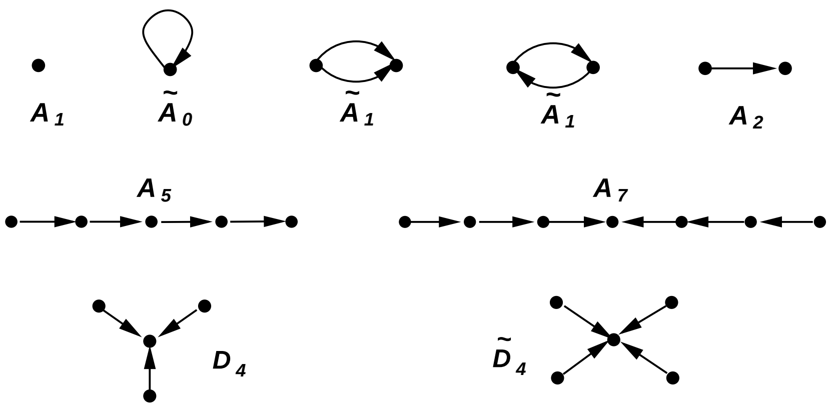

Figure 47

5. In $\mathbb{K}^n$, consider a pair of complete flags. They can be considered as a representation

$$\mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{n-1} \subset \mathbb{K}^n \supset \mathcal{W}^{n-1} \supset \cdots \supset \mathcal{W}^1$$

of the quiver, consisting of $2n-1$ vertices $v_1, \dots, v_n, v_{n+1} \dots v_{2n-1}$ (the case of $n = 4$ is shown on Figure 47 under the name A_7) connected by the edges $e_{12}, \dots, e_{n-1,n}$ and $e_{2n-1,2n-2}, \dots, e_{n+1,n}$.

6. A triple of subspaces: $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbb{K}^n$ can be considered as a representation of the quiver, denoted D_4 on Figure 47. We leave it for the reader to give examples of representations of quiver $\tilde{D}_4$, and to interpret representations of the two types of quivers $\tilde{A}_1$, shown on Figure 47.

Two representations of the same quiver are called **equivalent** if the spaces, corresponding to the vertices can be identified by isomorphisms in such a way that the corresponding maps are also identified. In greater detail, let $\mathcal{U}$ and $\mathcal{V}$ be two representations of the same quiver with vertices $\{v_i\}$. This means (see Figure 48, where the quiver is of type D_4, with a certain orientation of the edges) that we are given vector spaces $\mathcal{U}_i$ and $\mathcal{V}_i$, and for each edge e_{ij}, two linear maps: $A_{ij} : \mathcal{U}_i \to \mathcal{U}_j$, and $B_{ij} : \mathcal{V}_i \to \mathcal{V}_j$. To establish an equivalence between the representations, one needs to find *isomorphisms* $C_i : \mathcal{U}_i \to \mathcal{V}_i$ (shown on Figure 48 by vertical dashed arrows) such that the pairs of parallel horizontal and vertical arrows form **commutative squares**: $C_j A_{ij} = B_{ij} C_i$. In particular, corresponding spaces $\mathcal{U}_i, \mathcal{V}_i$ of equivalent quivers must have the same dimensions, and the corresponding maps A_{ij}, B_{ij} must have the same ranks.

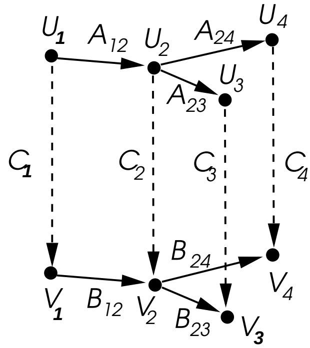

Figure 48

`Examples.` `7.` Two representations of the quiver A_1 (i.e. two vector spaces) are equivalent whenever the spaces are isomorphic. As we know, this this happens exactly when the spaces have the same dimension.

`8.` According to the Rank Theorem, two representations $A : \mathcal{U}_1 \to \mathcal{U}_2$ and $B : \mathcal{V}_1 \to \mathcal{V}_2$ of the same quiver $\bullet \to \bullet$ are equivalent whenever $\dim \mathcal{U}_i = \dim \mathcal{V}_i$ for $i = 1, 2$, and $\operatorname{rk} A = \operatorname{rk} B$. Indeed, when the dimensions n and m of the source and target space are fixed, and the rank $r \leq \min(m, n)$ is given, the representation is equivalent to the standard one: $E_r^{n,m} : \mathbb{K}^n \to \mathbb{K}^m$, given in coordinates by $(x_1, \dots, x_n) \to (x_1, \dots, x_r, 0, \dots, 0)$.

`9.` Two representations of the quiver of type $\tilde{A}_0$ are linear maps: $A : \mathcal{U} \to \mathcal{U}$, and $B : \mathcal{V} \to \mathcal{V}$. They are equivalent, whenever there is an isomorphism $C : \mathcal{U} \to \mathcal{V}$ such that $CA = BC$, i.e. $B = CAC^{-1}$. Since spaces of the same dimension n are isomorphic to $\mathbb{K}^n$, we conclude that classification of representations of this quiver up to equivalence coincides with the classification of square matrices up to similarity transformations. When $\mathbb{K} = \mathbb{C}$, the answer is given by the Jordan Canonical Form theorem, and for $\mathbb{K} = \mathbb{R}$ by the real version of this theorem.

As illustrated by above examples, each quiver leads to a well-posed classification problem of Linear Algebra: classification of representations of the given quiver up to equivalence. Moreover, two of the four classification theorems of Linear Algebra studied in this course turn out to answer such "quiver" problems corresponding to two very special examples: quivers A_2 (the Rank Theorem), and $\tilde{A}_0$ (the Jordan Canonical Form Theorem).

One way to classify representations of a given quiver is to reduce the problem to classification of *indecomposable representations.*

Given two representations of a given quiver, $\mathcal{U} = \{\mathcal{U}_i, A_{ij}\}$ and $\mathcal{V} = \{\mathcal{V}_i, B_{ij}\}$, one can form their **direct sum** $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ by taking $\mathcal{W}_i = \mathcal{U}_i \oplus \mathcal{V}_i$ on the role of the spaces, and $C_{ij} = A_{ij} \oplus B_{ij}$ on the role of the maps. The latter means that $C_{ij} : \mathcal{U}_i \oplus \mathcal{V}_i \to \mathcal{U}_j \oplus \mathcal{V}_j$ is defined by $C_{ij}(\mathbf{u}, \mathbf{v}) = (A_{ij}\mathbf{u}, B_{ij}\mathbf{v})$. In matrix terms, if linear maps A_{ij} and B_{ij} are given by their matrices in some bases of the respective spaces $\mathcal{U}_i, \mathcal{U}_j$ and $\mathcal{V}_i, \mathcal{V}_j$, then the matrix of C_{ij} is block-diagonal:

$$C_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & B_{ij} \end{bmatrix}.$$

A representation which is equivalent to the direct sum of non-zero[12] representations is called **decomposable**, and **indecomposable** otherwise.

`Example 10.` According to the Jordan Canonical Form Theorem, every representation of quiver $\tilde{A}_0$ over $\mathbb{K} = \mathbb{C}$ is equivalent to the direct sum of Jordan cells. Each Jordan cell is in fact *indecomposable.* Indeed, a Jordan cell has a one-dimensional eigenspace, but block-diagonal matrices have at least one one-dimensional eigenspace for each of the diagonal blocks.

`Example 11.` Each representation of quiver A_2: $\bullet \to \bullet$ is equivalent to the direct sum of three indecomposable representations:

$$\mathcal{U}^{01} : \mathbb{K}^0 \to \mathbb{K}^1, \quad \mathcal{U}^{11} : \mathbb{K}^1 \overset{\simeq}{\to} \mathbb{K}^1, \quad \mathcal{U}^{10} : \mathbb{K}^1 \to \mathbb{K}^0.$$

This follows from the Rank Theorem. Indeed, the matrix $E_r^{n,m} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is block-diagonal, with r diagonal blocks $\mathcal{U}^{11}$ and one *zero* diagonal block of size $(m-r) \times (n-r)$. The latter, i.e. the zero map from $\mathbb{K}^{n-r}$ to $\mathbb{K}^{m-r}$ can be described as the direct sum of $n-r$ copies of $\mathcal{U}^{10}$ and $m-r$ copies of $\mathcal{U}^{01}$.

[12] The **zero** representation has all $\mathcal{U}_i = \{\mathbf{0}\}$ and consequently all maps $A_{ij} = 0$.

Remark. A representation $\{\mathcal{U}_i, A_{ij}\}$ of a given quiver is called **reducible**, if there exists a non-trivial collection of subspaces $\mathcal{V}_i \subset \mathcal{U}_i$ which are respected by the maps A_{ij}, i.e. $A_{ij}(\mathcal{V}_i) \subset \mathcal{V}_j$. In this case, the subspaces $\mathcal{V}_i$ together with the restrictions of the maps A_{ij} to $\mathcal{V}_i$ also form a representation of the quiver: a **subrepresentation**. For instance, in a decomposable representation, each of the direct summands is a subrepresentation. A representation which does not have a non-trivial subrepresentation is called **irreducible**. For instance, $\mathcal{U}^{01}$, $\mathcal{U}^{11}$, and $\mathcal{U}^{10}$ are irreducible representations of quiver A_2. On the other hand, among Jordan cells $J_m : \mathbb{K}^m \to \mathbb{K}^m$, only the cell of size $m = 1$ is irreducible as a representation of $\tilde{A}_0$, because all Jordan cells of size $m > 1$ have non-trivial invariant subspaces. We see therefore two fundamental distinctions between the classification problems for the quivers A_2 and $\tilde{A}_0$: In the former case, there are finitely many (three) indecomposable representations, each of them is irreducible, and each representation is equivalent to a direct sum of them. In the latter case, there are infinitely many irreducible representations (they depend on parameters — eigenvalues), and indecomposable representations are not necessarily equivalent to a direct sum of irreducible ones.

Definition. A quiver is called **simple**, if indecomposable representations form finitely many equivalence classes.

Example 12. The quiver A_2 is simple, and $\tilde{A}_0$ is not.

Theorem (Pierre **Gabriel**[13]). ***A quiver is simple if and only if it has the form of one of the graphs*** $A_n, n \geq 1$, $D_n, n \geq 4$, $E_n, n = 6, 7, 8$ (see Figure 49), ***where*** n ***is the number of vertices, and the orientations of the edges are arbitrary.***

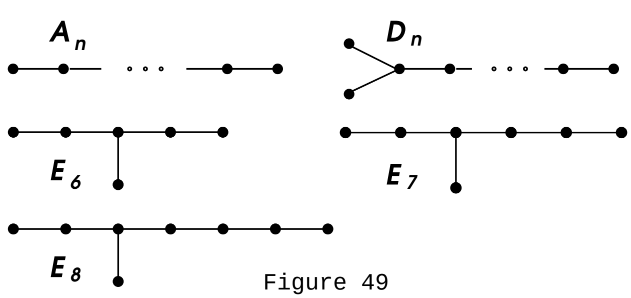

Figure 49

[13] P. Gabriel. *Unzerlegbare Darstellungen, I.* Manuscripta Mathematica 6 (1972), 71–103.

One statement of the theorem: that all these quivers are simple, is proved by classifying their indecomposable representations. This enthralling problem of linear algebra goes beyond our goal in this book.[14] Let us focus here on the converse statement: that only the quivers A_n, D_n, E_6, E_7, E_8 can be simple.

First, it should be clear that ***if a quiver is simple, then its representations in spaces of dimensions, not exceeding a certain bound, D, form finitely many equivalence classes.*** Indeed, let $d_1, \ldots, d_n$ denote the dimensions of the vector spaces $\mathcal{U}_i$ associated to the vertices $v_1, \ldots, v_n$ in a given representation $\mathcal{U}$. If this representation is equivalent to the direct sum of N indecomposable ones, then the total dimension $\sum d_i$ must be at least N. When $N > nD$, some d_i will exceed D. But if the number of types of indecomposable representations is finite, there are only finitely many ways to arrange them into direct sums of $\leq nD$ summands.

Let us now do some dimension count. Consider the problem of classification of representations of a given quiver, assuming that the dimensions $d_1, \ldots, d_n$ have been fixed. In other words, the space $\mathcal{U}_i$ associated to the vertex v_i can be identified with the coordinate space $\mathbb{K}^{d_i}$ by a choice of basis. The operator $A_{ij} : \mathcal{U}_i \to \mathcal{U}_j$ corresponding to an edge e_{ij} is then described (in the chosen bases of the spaces $\mathcal{U}_i$ and $\mathcal{U}_j$) by an $d_j \times d_i$-matrix (which can be arbitrary!) Thus, all representations with prescribed dimensions $d_1, \ldots, d_n$ of the spaces form themselves a vector space of the total dimension $\sum_{\text{edges } e_{ij}} d_i d_j$ (this is how many entries the matrices A_{ij} have).

Thus, a representation is specified by a collection $\{A_{ij}\}$ of matrices of prescribed sizes $d_j \times d_i$, one for each edge, but some such collections define *equivalent* representations. Which ones? The representation defined by the collection $\{A'_{ij}\}$ is equivalent to the previous one, if there exist invertible transformations $C_i : \mathcal{U}_i \to \mathcal{U}_i$ such that $A'_{ij} = C_j A_{ij} C_i^{-1}$ for each edge e_{ij}. In other words, equivalent representations are obtained from each other by bases changes in the spaces $\mathcal{U}_i = \mathbb{K}^{d_i}$. Such bases changes are determined by n invertible matrices C_i of sizes $d_i \times d_i$, which depend therefore on $\sum_{\text{vertices } v_i} d_i^2$ parameters. In fact one parameter here is "wasted": if all matrices C_i are equal to the same non-zero scalar λ (i.e. $C_i = \lambda I_{d_i}$), then $A'_{ij} = A_{ij}$. We conclude that, ***the number of parameters,***

[14] We refer the reader to the paper: I. N. Bernten, I. M. Gelfand, V. A. Ponomarev, *Coxeter functors, and Gabriel's theorem.* Russian mathematical surveys 28, No. 2, (1973), 1732., where an illuminating proof of Gabriel's theorem is given.

needed to parameterize equivalence classes of representations with prescribed dimensions $d_1, \dots, d_n$ ***of the spaces, is greater than*** $\sum_{\text{edges } e_{ij}} d_i d_j - \sum_{\text{vertices } v_i} d_i^2$.

EXERCISES

418. Interpret representations of the two quivers $\tilde{A}_1$ (Figure 47) in matrix terms. ✓

419. On the plane, consider three distinct lines passing through the origin as a representation of quiver D_4 and show that they form a single equivalence class. ↯

420. Consider four distinct lines on the plane as a representation of quiver $\tilde{D}_4$, and show that the equivalence classes depend on one parameter. ↯

421.* Prove that the **cross-ratio** $\lambda := (a-b)(c-d)/(a-c)(b-d)$ of the slopes a, b, c, d of four lines on the plane does not change under linear transformations of the plane. ↯

422. Find all indecomposable representations of quiver A_1. ✓

423.* Show directly that equivalence classes of some representations of quiver $\tilde{A}_n$ (with any orientation of the edges) depend on at least one parameter. ↯

424. Consider a Jordan cell of size 2 as a representation of quiver $\tilde{A}_0$, and find all nontrivial subrepresentations. ✓

425. Show that all complete flags in $\mathbb{K}^n$ considered as representations of quiver A_n are equivalent.

426. Show that pairs of complete flags in $\mathbb{K}^n$ considered as representations of quiver A_{2n-1} form $n!$ equivalence classes. ↯

Graphs and Quadratic Forms

To a graph Γ with n vertices $v_1, \dots, v_n$ and edges $\{e_{ij}\}$, we associate quadratic form Q_Γ on the space $\mathbb{R}^n$ with coordinates $x_1, \dots x_n$ by the formula

$$Q_\Gamma(\mathbf{x}) := \sum_{\text{vertices } v_i} 2x_i^2 - \sum_{\text{edges } e_{ij}} 2x_i x_j.$$

`Example 13.` When Γ is the graph A_n, the quadratic form is $2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + \cdots + 2x_{n-1}^2 - 2x_{n-1}x_n + 2x_n^2$. The quadratic form corresponding to $\Gamma = \tilde{A}_0$ is $2x_1^2 - 2x_1x_1 = 0$ (identically zero!)

`Proposition.` ***If a quiver*** Γ ***is simple, then the quadratic form*** Q_Γ ***is positive definite.***

`Proof.` We argue *ad absurdum.* Suppose that $\mathbf{x} \neq 0$, and $Q_\Gamma(\mathbf{x}) \leq 0$. We may assume that all components of $\mathbf{x} = (x_1, \ldots, x_n)$ are rational numbers. Indeed, if Q_Γ is non-negative, but not positive definite, then the coefficients matrix Q_Γ is degenerate. Since it has integer coefficients, the system $Q_\Gamma\mathbf{x} = \mathbf{0}$ has non-trivial *rational* solution. Alternatively, if $Q_\Gamma(\mathbf{x}_0) < 0$ for some $\mathbf{x}_0$, one can approximate $\mathbf{x}_0$ with a rational vector $\mathbf{x}$ and still have $Q_\gamma(\mathbf{x}) < 0$. Moreover, by clearing denominators, we may assume that all x_i are integers. Note that replacing all x_i with $|x_i|$ can only decrease the value $Q_\Gamma(\mathbf{x})$, because the terms x_i^2 do not change, but the terms $-x_ix_j$, if change at all, then from a positive value to the opposite negative one. Thus, unless Q_Γ is positive definite, there exist a vector $\mathbf{d} = (d_, \ldots, d_n)$ of non-negative integers not all equal to 0 and such that $Q_\Gamma(\mathbf{d}) \leq 0$. Therefore equivalence classes of representations of the quiver Γ with the dimensions of the spaces equal to $d_1, \ldots, d_n$ (and any orientation of the edges) depend on at least one parameter (on more than $-Q_\Gamma(\mathbf{d})/2$ parameters, to be more precise). Thus the number of such equivalence classes is infinite (since $\mathbb{K}$ is), and the quiver is not simple.

`Theorem.` ***The quadratic form*** Q_Γ ***of a graph*** Γ ***is positive definite if and only if each connected component of the graph is one of:*** $A_n, n \geq 1$***,*** $D_n, n \geq 4$***,*** E_6, E_7, E_8 (Figure 49).

We will prove this theorem in several steps.

Put $\Delta(\Gamma) := \det Q_\Gamma$, the determinant of the coefficient matrix of the quadratic form Q_Γ.

`Proposition.`

$$\Delta(A_n) = n + 1,\ \ \Delta(D_n) = 4,\ \ \Delta(E_6) = 3,\ \ \Delta(E_7) = 2,\ \ \Delta(E_8) = 1.$$

`Lemma.` *Let* v_1 *be a vertex of* Γ *connected by one edge with the rest of the graph,* Γ'*, at the vertex* v_2*, and let* Γ'' *be the graph obtained from* Γ' *by removing the vertex* v_2 *together with all the edges attached to it. Then* $\Delta(\Gamma) = 2\Delta(\Gamma') - \Delta(\Gamma'')$*.*

Indeed, $\det Q_\Gamma$ looks this way, where $\mathbf{0}$ is a row/column of zeroes, and $*$ is a (row/column) of "wild cards":

$$\begin{vmatrix} 2 & -1 & \mathbf{0} \\ -1 & * & * \\ \mathbf{0} & * & Q_{\Gamma''} \end{vmatrix}.$$

Applying the cofactor expansion in the 1st row, and then in the 1st column, we find $\Delta(\Gamma) = 2\Delta(\Gamma') - (-1)^2\Delta(\Gamma'')$, as required. $\square$.

Now, to prove the proposition for $\Gamma = A_n$, we use induction on n. Namely, $\Delta(A_1) = 2$, $\Delta(A_2) = 3$, and from the induction hypothesis $\Delta(A_{n-1}) = n$, $\Delta(A_{n-2}) = n-1$, we conclude using Lemma:

$$\Delta(A_n) = 2\Delta(A_{n-1}) - \Delta(A_{n-2}) = 2n - (n-1) = n+1$$

as required.

For $\Gamma = D_n, E_n$, we apply Lemma to the vertex v_2 with 3 edges, and take v_1 to be the end of the shortest leg. The graph Γ'' falls apart into two components Γ_1, Γ_2. The corresponding matrix is block-diagonal, and the determinant factors: $\Delta(\Gamma'') = \Delta(\Gamma_1)\Delta(\Gamma_2)$. Thus:

$$\begin{aligned}
\Delta(D_n) &= 2\Delta(A_{n-1}) - \Delta(A_1)\Delta(A_{n-3}) = 2n - 2(n-2) = 4,\\
\Delta(E_6) &= 2\Delta(A_5) - \Delta(A_2)\Delta(A_2) = 2\cdot 6 - 3\cdot 3 = 3,\\
\Delta(E_7) &= 2\Delta(A_6) - \Delta(A_2)\Delta(A_3) = 2\cdot 7 - 3\cdot 4 = 2,\\
\Delta(E_8) &= 2\Delta(A_7) - \Delta(A_2)\Delta(A_4) = 2\cdot 8 - 3\cdot 5 = 1. \quad \square
\end{aligned}$$

`Corollary 1.` ***For*** $\Gamma = A_n, D_n, E_6, E_7, E_8$***, the quadratic form*** Q_Γ ***is positive definite.***

`Proof.` This follows from Sylvester's rule. Order somehow the vertices of the graph Γ, and tear them off one by one (together with the attached edges). On each step, we obtain a graph Γ' which is one of the graphs A_k, D_k, E_k, or a collection thereof. By Proposition, $\Delta(\Gamma') > 0$. This means that all leading minors of the coefficient matrix Q_Γ are positive, and hence the quadratic form is positive definite. $\square$

`Corollary 2.` ***For each of the graphs*** $\tilde{\Gamma} := \tilde{A}_n, n \geq 0$***,*** $\tilde{D}_n, n \geq 4$***,*** $\tilde{E}_n$***,*** $n = 6, 7, 8$ ***shown on Figure 50, the corresponding quadratic form*** $Q_{\tilde{\Gamma}}$ ***on*** $\mathbb{R}^{n+1}$ ***is non-negative. More precisely, its positive inertia index equals*** n***, and the 1-dimensional kernel is spanned by the vector whose components are shown on the diagram.***

`Proof.` By tearing off $\tilde{\Gamma}$ the "white" vertex together with the attached edges, we obtain the corresponding graph Γ, whose quadratic form Q_Γ is positive definite by Corollary 1. Thus, $\mathbb{R}^{n+1}$ contains the subspace $\mathbb{R}^n$ on which the quadratic form $Q_{\tilde{\Gamma}}$ is positive definite, which proves that the positive inertia index is at least n. To prove that the quadratic form $Q_{\tilde{\Gamma}}$ is degenerate, consider the corresponding symmetric bilinear form

$$Q_{\tilde{\Gamma}}(\mathbf{x}, \mathbf{y}) = \sum_{\text{vertices } v_i} (2x_i - \sum_{\text{edges } e_{ij}} x_j) y_i,$$

where the last sum is taken over *all* edges attached to the vertex v_i. To show that a vector $\mathbf{x}$ is Q-orthogonal to every $\mathbf{y}$, it suffices to check that for each vertex v_i, twice the value x_i is equal to the sum of x_j over all vertices v_j connected to v_i: $2x_i = \sum_{\text{edges } e_{ij}} x_j$. It is straightforward to check that this requirement holds true for the positive integers written next to the vertices on the diagrams of Figure 50. Thus, the vector with these components lies in the quadratic form, which also shows that the negative inertia index of the quadratic form must be 0. □

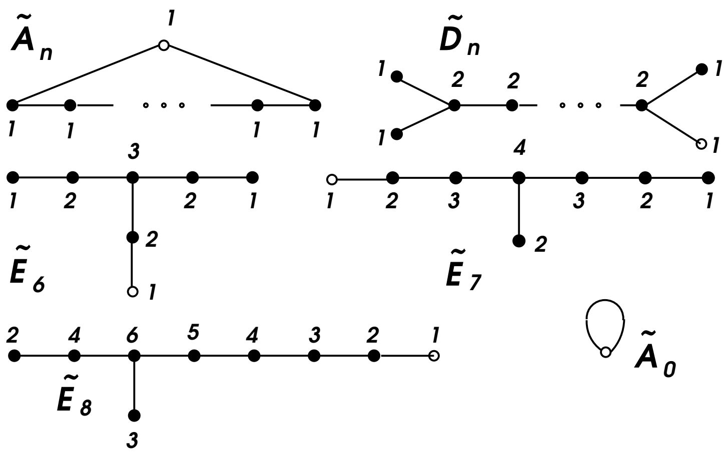

Figure 50

Corollary 2 shows that the quadratic form Q_Γ cannot be positive definite if Γ contains any of the graphs of Figure 50 as a subgraph. (By a *subgraph* of Γ we mean some vertices of Γ connected by *some* of the edges of Γ.) Indeed, the labels on Figure 50 exhibit a vector with non-negative components on which Q_Γ is non-positive. For, adding extra edges can only decrease the value of the form, and the effect of adding extra vertices can be offset by putting $x_i = 0$ at these vertices. Thus, to complete the proof of our Theorem, it remains to show that ***a connected graph*** Γ***, free of subgraphs*** $\tilde{A}_n, n \geq 0$***,*** $\tilde{D}_n, n \geq 4$***, and*** $\tilde{E}_n, n = 6, 7, 8$***, must be one of the graphs*** $A_n, n \geq 1$***,*** $D_n, n \geq 4$***,*** $E_n, n = 6, 7, 8$***.***

To justify the highlighted claim, note that Γ cannot contain a *loop* (i.e. a vertex connected by an edge with itself), nor a *cycle* (for, $\tilde{A}_n$ are exactly these). A connected graph free of loops or cycles is called a *tree*. Thus Γ is a tree. This tree cannot contain a vertex with ≥ 4 edges attached (for, $\tilde{D}_4$ is just that). Nor it can contain two vertices with 3 edges attached to each of them (for, a subgraph $\tilde{D}_n$ with $n > 4$ will be found by connecting these vertices with a chain of edges).

If Γ has no vertices with 3 edges attached, then it is of type A_n. If it has one such a vertex, then $\Gamma = T_{p,q,r}$, i.e. it has the shape of a letter "T" with the legs of type A_p, A_q, and A_r (where p, q, r are integers > 1) connected at a common vertex. If $p, q, r \geq 3$, then Γ contains $\tilde{E}_6$ as a subgraph. If $p = 2$, but $q, r \geq 4$, then Γ contains $\tilde{E}_7$ as a subgraph. If $p = 2, q = 3$, but $r \geq 6$, then Γ contains $\tilde{E}_8$ as a subgraph. Thus, assuming $p \leq q \leq r$, we have only the following options left: $(p, q, r) = (2, 3, 5)$, $(2, 3, 4)$, $(2, 3, 3)$, which yield the graphs E_8, E_7, E_6, or $p = q = 2$ (while $r \geq 2$ can be arbitrary), which yields the graphs D_{r+2}. Theorem is proved.

EXERCISES

427.* Prove that $\Delta(T_{p,q,r}) = pq - qr + rp - pqr$. ϟ

428. Show that $Q_{T_{p,q,r}}$ is positive definite (respectively, non-negative) if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (respectively $= 1$). ϟ

429.* Find all integer triples (p, q, r), $2 \leq p \leq q \leq r$, satisfying
(a) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$; (b) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. ✓

430. Tile the Euclidean plane by congruent triangles with the angles:
(a) $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, (b) $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, (c) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$.

431.* Tile the sphere by spherical triangles with the angles:
(a) $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{r})$; (b) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$, (c) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})$, (d) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5})$. ϟ

Root Systems

A remarkable feature of the classification by the graphs A_n, D_n, E_n is that they arise not only in connection with quivers, or graphs Γ with positive definite quadratic forms Q_Γ, but in a myriad of other classification problems. Namely, they occur in the theory of: regular polyhedra in the 3-space, reflection groups in Euclidean spaces, compact Lie groups, degenerations of functions near critical points, singularities of wave fronts and caustics of geometrical optics, and perhaps many others. There exist many direct connections between different manifestations of the ADE-classification, but the general cause of the phenomenon remains a mystery. We round up these notes with one illustration to the mystery, which comes from the theory of reflection groups.

Given a graph Γ, one can consider the symmetric bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle = (Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}))/2$ associated with the quadratic form $Q = Q_\Gamma$ as an inner product in $\mathbb{R}^n$ in a generalized sense (for it is not guaranteed to be positive definite). Let $\mathbf{v}_i$ be the vectors of the standard basis in $\mathbb{R}^n$. They correspond to the vertices of the graph and have length $\sqrt{2}$, i.e. $Q(\mathbf{v}_i) = 2$ (assuming that Γ has no loops attached to the ith vertex). One can associate to Γ the *group* G_Γ generated by n reflections in the hyperplanes Q-orthogonal to $\mathbf{v}_i$.

In more detail, by a **group** of Q-orthogonal transformations one means a collection $G = \{U_\alpha\}$ of transformations $U_\alpha : \mathbb{R}^n \to \mathbb{R}^n$, preserving the inner product, i.e. satisfying $\langle U_\alpha \mathbf{x}, U_\alpha \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and such that the compositions $U_\alpha U_\beta$ and inverses U_α^{-1} of the transformations from the collection G are also in G.

For instance, to each vector $\mathbf{v} \in \mathbb{R}^n$ with $Q(\mathbf{v}) = 2$ one can associate the **reflection** $R_\mathbf{v}$ in the hyperplane Q-orthogonal to $\mathbf{v}$:

$$R_\mathbf{v}\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v}.$$

If $\langle \mathbf{x}, \mathbf{v} \rangle = 0$, then $R_\mathbf{v}\mathbf{x} = \mathbf{x}$, while $R_\mathbf{v}\mathbf{v} = \mathbf{v} - 2\mathbf{v} = -\mathbf{v}$. Thus $\mathbb{R}^n$ is the direct Q-orthogonal sum of two eigenspaces of $R_\mathbf{v}$ corresponding to the eigenvalues 1 and -1, and so $R_\mathbf{v}$ preserves Q.

Given several vectors $\{\mathbf{v}_i\}$ with $Q(\mathbf{v}_i) = 2$, one can generate a group of Q-orthogonal transformations by considering all the reflections $R_{\mathbf{v}_i}$ along with their compositions, inverses, compositions of the compositions, etc.

It turns out that ***the reflection group*** G_Γ ***associated with a graph*** Γ ***without loops is finite if and only if each connected component of*** Γ ***is one of the graphs*** A_n, D_n, E_6, E_7, E_8***.*** We are not going to prove this theorem here (although this would not be too hard to do by checking that if Γ contains $\tilde{A}_n, \tilde{D}_n$, or $\tilde{E}_n$, then the reflection group must be infinite), but merely illustrate it with a useful example.

`Example 14:` *Reflection group* A_n. In the standard Euclidean space $\mathbb{R}^{n+1}$ with coordinates $x_0, \ldots, x_n$, consider the hyperplane given by the linear equation $x_0 + \cdots + x_n = 0$. Permutation of the coordinates form a group of $(n+1)!$ orthogonal transformations in $\mathbb{R}^{n+1}$, preserving the hyperplane. The whole group of permutations is generated by n transpositions $\tau_{01}, \tau_{12}, \ldots, \tau_{n-1,n}$, where τ_{ij} stands for the swapping of the coordinates x_i and x_j, and thus acts as an orthogonal reflection in the hyperplane $x_i = x_j$.

The hyperplane $x_0 + \cdots + x_n = 0$ can be identified with $\mathbb{R}^n$ by the choice of a basis:

$$\mathbf{v}_1 = \mathbf{e}_0 - \mathbf{e}_1,\ \mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2,\ \ldots,\ \mathbf{v}_n = \mathbf{e}_{n-1} - \mathbf{e}_n,$$

where $\mathbf{e}_i = (\ldots, 0, 1, 0, \ldots)^t$ denote the unit coordinate vectors in $\mathbb{R}^{n+1}$. Computing pairwise dot-products, we find:

$$\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 2,\ \langle \mathbf{v}_i, \mathbf{v}_{i+1} \rangle = -1,\ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \text{ for } |i-j| > 1.$$

We see that the Euclidean structure on the hyperplane $\mathbb{R}^n$ in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ coincides with the one defined by the graph A_n. Note that the reflections in the hyperplanes perpendicular to $\mathbf{v}_i$ are exactly the transpositions $\tau_{i-1,i}$. Thus, the reflection group G_{A_n} is identified with the group of permutations of $n+1$ objects.

There is more here than meets the eye. While the group is *generated* by n reflections, the total number of hyperplane reflections in it is $\binom{n+1}{2} = n(n+1)/2$: one for each transposition τ_{ij}. Respectively there are $n(n+1)$ vectors $\mathbf{v}$ of length $\sqrt{2}$ perpendicular to the hyperplanes: $\pm(\mathbf{e}_i - \mathbf{e}_j)$. The configuration of these vectors (called "roots") is called the **root system** associated to the graph A_n. The same can be done with each of the A, D, E-graphs: the root system is a finite symmetric configuration of vectors of length $\sqrt{2}$ obtained from any of the vectors $\mathbf{v}_i$ by applying all transformations from the reflection group. One of many peculiar properties of root systems is that *each root $\mathbf{v}$ is a linear combination of the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (corresponding to the vertices of the graph) with coefficients which have the same sign: either all non-negative, or all non-positive.* E.g. in the case A_n, if $i < j$, then $\pm(\mathbf{e}_i - \mathbf{e}_j) = \pm(\mathbf{v}_{i+1} + \cdots + \mathbf{v}_j)$. Thus, all the roots are divided into positive (including $\mathbf{v}_1, \ldots, \mathbf{v}_n$) and negative.

We can now state the following addition to Gabriel's theorem: ***Indecomposable representations of a simple quiver are in one-to-one correspondence with positive roots of the corresponding root system, and the components $d_i \geq 0$ of a positive root $\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n$ are the dimensions of the spaces of the corresponding indecomposable representation.***

`Example 15:` A_2. The reflection group G_{A_2} acts on the plane $x_0 + x_1 + x_3 = 0$ by symmetries of a regular triangle. There are three symmetry axes, and respectively 6 roots perpendicular to them, three of which are positive: $\mathbf{v}_1 = \mathbf{e}_0 - \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{v} = \mathbf{e}_0 - \mathbf{e}_2 = \mathbf{v}_1 + \mathbf{v}_2$. Their coefficients $(1,0)$ $(0,1)$, and $(1,1)$ correspond to the three indecomposable representations of the quiver $\bullet \to \bullet$ as described by the Rank Theorem: $\mathbb{K}^1 \to \mathbb{K}^0$, $\mathbb{K}^0 \to \mathbb{K}^1$, and $\mathbb{K}^1 \overset{\simeq}{\to} \mathbb{K}^1$.

EXERCISES

432. Verify that reflections $R_{\mathbf{v}}$ are Q-orthogonal. ✓

433.* Prove that the reflection group G_Γ corresponding to the graph $\Gamma = \tilde{A}_1$ is infinite. ↯

434. Describe all indecomposable representations of quiver A_n: $\bullet \to \bullet \to \cdots \to \bullet \to \bullet$. ↯

435. Represent a complete flag in $\mathbb{K}^n$ as the direct sum of indecomposable representations of quiver A_n. ✓

436. Let X denote an $n \times n$-matrix. Find the solution to the ODE system $dX/dt = AX - XA$ (where X stands for the unknown $n \times n$-matrix), given the intial value $X(0)$. ✓

437. In the previous exercise, let A be diagonal, with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvectors and eigenvalues of the operator $X \mapsto AX - XA$ on the space of $n \times n$-matrices, and compare the answer with the root system of type A. ✓