Math 242. HW Solutions

1. Let $e_1,\dots,e_r,f_1,\dots,f_r$ be vectors in a given subspace $W^{2r+k}\subset(\mathbb{R}^{2n},\omega)$ with $\mathrm{rk}(\omega|_W)=2r$, which project to a Darboux basis of the quotient $W/\ker(\omega|_W)$. They span a symplectic subspace $W'\subset W\subset\mathbb{R}^{2n}$, whose skew-orthogonal complement W'' is a complementary symplectic subspace of dimension $2n-2r$ containing isotropic $W_0:=\ker(\omega|_W)$. Let $e_{r+1},\dots,e_{r+k}$ be a basis in W_0, and $f_{r+1},\dots,f_{r+k}$ be such vectors in W'' for which the linear forms $\omega(f_i,\cdot)$, when restricted to W_0 form the basis in W_0^* dual to $\{e_i|i=r+1,\dots r+k\}$. Then $\{e_i,f_i|i=r+1,\dots,r+k\}$ is a Darboux basis in a sumplectic subspace in W'' of dimension $2k$. Let W''' be its skew-orthogonal complement in W'', and $\{e_i,f_i|i=r+k+1,\dots,n\}$ be a Darboux basis in W'''. Then $\{e_if_i|i=1,\dots n\}$ form a Darboux basis in $\mathbb{R}^{2n}$, and W is spanned by $e_1,\dots,e_{r+k},f_1,\dots,f_r$ as required.

3. For $\phi=\sum_{1\le i<j\le 4}\phi_{ij}x_i\wedge x_j$, the Plücker relation $\phi\wedge\phi=0$ has the form $2(\phi_{12}\phi_{34}-\phi_{13}\phi_{24}+\phi_{14}\phi_{23})=0$. Since $4uv=(u+v)^2-(u-v)^2$, this quadratic relation can be rewritten in new coordinates as $a_1^2+a_2^2+a_3^2-b_1^2-b_2^2-b_3^2=0$. The projective space $\mathbb{R}P^5$ can be identified with the quotient of the sphere $a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2=2$ by the central symmetry $(a,b)\mapsto(-a,-b)$. The Plücker relation cuts out in $\mathbb{R}P^5$ the hypersurface $Gr_{2,4}$, which is thereby identified with the product $S^2\times S^2=\{(a,b)||a|=1=|b|\}$ by this involution. Furthermore, the restriction of a quadratic form of signature $(+++---)$ to a hyperplane, if remains non-degenerate, has the signature $(+++--)$ or $(++---)$. The zero cone of either form cuts out in $\mathbb{R}P^4$ a hypersurface identified (similarly to the previous argument) with the quotient of $S^2\times S^1$ by the simultaneous central symmetry $\sigma:(a,b)\mapsto(-a,-b)$. Thus the Lagrange grassmannian $\Lambda_2\cong(S^2\times S^1)/\sigma$.

10. The equation of non-linear pendulum is $\ddot\theta=-\omega^2\sin\theta$, where $\omega^2=g/l$, linearizes near $\theta=\pi$ to $\ddot x=+\omega^2x$. The Hamiltonian (total energy) $H=(\dot x^2-\omega^2x^2)/2$ has a saddle-like critical point at the origin (the phase portrait is formed by the level curves of H). In rescaled Darboux coordinates $q=\sqrt{\omega}x$, $p=\dot x/\sqrt{\omega}$, $H=\omega(p^2-q^2)/2$, and the phase flow is described by the 1-parametric group $\begin{bmatrix}\cosh\omega t & \sinh\omega t\\ \sinh\omega t & \cosh\omega t\end{bmatrix}$ of hyperbolic rotations.

13. In fact it seems easier to answer first the 2nd question (related to the partitions of the parameter spaces into orbits) by general reasoning, and then exhibit miniveral deformations. In $\mathbb{C}^2$, consider the Hermitian form(s) with the real part $\langle z,w\rangle_\pm=z_1\bar w_1\pm z_2\bar w_2$. For either sign, the imaginary part is a symplectic form on the realification of $\mathbb{C}^2$, while the real part $|z_1|^2 \pm |z_2|^2$ is the quadratic hamiltonian whose flow (if followed during suitable time — perhaps $\pi/2$) defines the multiplication by $\sqrt{-1}$. Consequently the group of symplectic automorphisms preserving the hamiltonian automatically preserves the compmex structure, and coincides with the group of unitary automorphisms of the Hermitian form: $U(2)$ for the sign $+$, and $U(1,1)$ for $-$. In particular, the codimension of the adjoint orbit of either hamiltonian is equal to 4 (the dimension of the stabilizer). Knowing this, let us write down miniveral deformations of the hamiltonian(s) $h_\pm = (p_1^2+q_1^2) \pm (p_2^2+q_2^2)$. The matrix of the correpsponding hamiltonian operator $H_\pm$ turns out anti-symmetric, $H_\pm^* = -H_\pm$ and so infinitesimal symplectic transformations commuting with $H_\pm^*$ are the same as those commuting with $H_\pm$ itself. It is not too hard to guess 4 independent quadratic hamiltonians f Poisson-comuting with $h_\pm$, i.e. satisfying:

$$(p_1\partial f/\partial q_1 - q_1\partial f/\partial p_1) \pm (p_2\partial f/\partial q_2 - q_2\partial f\partial p_2) = 0.$$

Indeed, $f = p_1^2+q_1^2, p_2^2+q_2^2, p_1q_2 \mp q_1p_2, p_1p_2 \pm q_1q_2$ will do (as it is easy to check). Thus a miniversal deformation of $h_\pm$ can be written as the germ at $(\lambda, \mu, \xi, \eta) = (1,1,0,0)$ of the family

$$\lambda(p_1^2+q_1^2) \pm \mu(p_2^2+q_2^2) + \xi(p_1q_2 \mp q_1p_2) + \eta(p_1p_2 \pm q_1q_2).$$

After a difect computation, I found that the characteristic polynomial of the corresponding hamiltonian operator equals

$$t^4 + 4\left(\lambda^2+\mu^2 \pm \frac{\xi^2+\eta^2}{2}\right)t^2 + (\xi^2+\eta^2 \mp 4\lambda\mu)^2$$

with the discriminant of the quadratic equation in t^2 equal to

$$16(\lambda+\mu)^2[(\lambda-\mu)^2 \pm (\xi^2+\eta^2)].$$

Since $\lambda, \mu \approx 1$, multiple eigenvalues occur only when $(\lambda-\mu)^2 = \mp(\xi^2+\eta^2)$, hence do not occur in the "+" case unless $\xi = \eta = 0, \lambda = \mu$, i.e. for hamiltonians proportional to h_+. As it was clear from the very beginning, since h_+ is positive definite, so are all nearby hamiltonians, and all of them are stable. In the "$-$" case, $(\lambda-\mu)^2 = \xi^2+\eta^2$ is the equation in $\mathbb{R}^4$ of a cylinder over a 3-dimensional cone. The "axis" $\lambda = \mu, \xi = \eta = 0$ of this cylinder corresponds to hamiltonians proportional to h_-, which are stable, but not strongly stable. Indeed, when $(\lambda-\mu)^2 < \xi^2+\eta^2$, i.e. the discrimiant is negative, the values of t^2 are non-real, and hence the values of t form a quadruple of two pairs of complex-conjugated eigen-values — one with negative and one with positive real part. The latter indicate instability.

All of this could be found, however without much computation. Namely, our 4-dimensional family is in fact a Lie subalgebra in $sp(4,\mathbb{R})$, isomorphic, as we have concluded earlier, to $u(2)$ in the "+" case and respectively $u(1,1)$ in the "−" case. It consists of matrices of the form $\begin{bmatrix} i\alpha+i\beta & \mp\bar{z} \\ z & i\alpha-i\beta \end{bmatrix}$, with the characteristic polynomial $(t-i\alpha)^2+(\alpha^2+\beta^2\pm|z|^2$. In the "+" case, the roots are imaginary (so, all transformations are stable), and in the "−" case, those with $\alpha^2+\beta^2<|z|^2$ have non-imaginary eigenvalues, and those with $\alpha^2+\beta^2=|z|^2$ are not diagonalizable unless they are scalar (i.e. unless $\beta=0, z=0$).

Coincidentally or not, the partition of the parameter space into equivalence classes of symplectic transformations coincides with the partition of the Lie algebra ($u(2)$ or $u(1,1)$) into its own adjoint orbits. In the $u(2)$ case, they fill parallel 3-dimensional subspaces (the level sets of the trace function) into concentric spheres (centered at the scalar matrix with the given trace). In the $u(1,1)$ case, they fill parallel 3-dimensional level sets of the trace function with the 1- and 2-sheeted hyperboloids (and a cone with the vertex at the scalar matrix with the prescribed trace). The latter picture coincides with the partition of $sp(2,\mathbb{R})$ into adjoint orbits — not surprisingly though, since $sp(2,\mathbb{R})\cong sl(2,\mathbb{R})\cong su(1,1)$ as Lie algebras.

20. We are looking for a family of origin-preserving diffeomorphisms g_t, $g_0=\mathrm{Id}$, such that $f_t(g_t(x))=f_0$, where f_0 is the quadratic form $d_0^2 f$ (one may assume it is $\sum \pm x_i^2$), and $f_t=f_0+t(f-f_0)$. Diferentiating in t, we obtain $(L_{v_t}f_t)(g_t(x))+(f-f_0)(g_t(x))=0$, where $x(t)=g_t(x(0))$ is to be defined by solutions of the ODE system $dx/dt=v_t(x)$. More explicitly, we need to find the components $v_t^{(i)}$ of the vector field v_t such that $\sum v_t^{(i)}\partial f_t/\partial x_i=f_0-f$, where the RHS is given. What helps is that $\partial f_t/\partial x_i$ can be taken for new (time-dependent) coordinates, y_i. The RHS, which is $o(|x|^2)$, can be expressed in these new coordinates as some $\phi_t(y)$, and then expanded as $\phi_t(y)=\sum_i w_t^{(i)}(y)y_i$ using Hadamard's lemma. The components $v_t^{(i)}(x)$ of the vector field $v_t(x)$ are obtained from $w_t^{(i)}(y)$ by reversing our change of coordinates from x to y, and come out as $o(|x|)$. Consequently the diffeomorphosms g_t come out not only preserving the origin, but also identical on the tangent space $T_0\mathbb{R}^n$. To guarantee smooth time dependence of the solution $w_t^{(i)}(y)$, let us recall the proof of Hadamard's lemma:

$$\phi(0)-\phi(y)=\int_0^\infty \frac{d}{du}\phi(e^{-u}y)du=\sum_i y_i\int_0^\infty \frac{\partial\phi}{\partial y_i}(e^{-u}y)e^{-u}du,$$

where the integrals on the right clearly depend smoothly on parameters (such as t) should such parameters be present in the function ϕ.

23. The group of affine transfromations on the line can be identified with the group of matrices $G = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a \neq 0$, and its Lie algebra with that of matrices $X = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$. We have

$$GXG^{-1} = \begin{bmatrix} ax & ay \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & ay - bx \\ 0 & 0 \end{bmatrix}.$$

Thus, the adjoint orbits are: (i) the origin $(x, y) = (0, 0)$, (ii) the "punctured" line $x = 0, y \neq 0$, and (iii) every line $x = const \neq 0$.

The dual of the Lie algebra can be identified with the space of matrices $\Theta = \begin{bmatrix} \mu & 0 \\ \nu & 0 \end{bmatrix}$ using the pairing $\langle X, Y \rangle := \operatorname{tr} X\Theta = x\mu + y\nu$. Consequently the coadjoint action (of G^{-1}) is given by $\operatorname{tr} GXG^{-1}\Theta = x\mu + (ay - bx)\nu$, i.e. $Ad^*_{G^{-1}}(\mu, \nu) = (\mu - b\nu, a\nu)$ where $a \neq 0$. Thus, the coadjoint orbits are (i) every point $(\mu, 0)$, and (ii) the rest of the plane, $\{(\mu, \nu) \mid \nu \neq 0\}$.

24. It is not hard to get the solution directly from the definitions, but let us exploit our facility with Poisson structures. The coordinates x, y, z on the dual of our Lie algebra are generators of the Lie algebra itself, and the Poisson brackets among them are to follow the definition of the cross-product in an orthonormal basis: $\{x, y\} = z$, $\{y, z\} = x$, $\{z, x\} = y$. With $F = (x^2 + y^2 + z^2)/2$, we have (from the Leibniz rule):

$$\{x, F\} = \{x, x\}F_x + \{x, y\}F_y + \{x, z\}F_z = 0x + zy - yz = 0,$$

and likewise, $\{y, F\} = \{z, F\} = 0$, implying that F is a Casimir function, i.e. is constant on symplectic leaves. Since the rank of the Poisson structure outside the origin equals 2, the level sets are the symplectic leaves (a.k.a. coadjoint orbits). The Poisson tensor is $W = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y$, so we have $i_W(dx \wedge dy \wedge dz) = xdx + ydy + zdz = dF$, implying that the contraction between W and the Leray form equals 1, which on 2-dimensional leaves, means that the Leray 2-form is the tensor field inverse to the Poisson tensor.

`Remark.` Actually it was clear *a priori* that the coadjoint orbits are concentric spheres, and that the symplectic forms on them are rotationally-invariant. So, the problem was about the correct normalization of the area form on the shere of radius r. Note that the Leray form has physical dimension $[inches]$ and therefore differs from both the Euclidean area form $r^2 \sin\phi d\phi \wedge d\theta$, (whose dimention is $[inches]^2$) and $i_{x\partial_x + y\partial_y + z\partial_z} dx \wedge dy \wedge dz = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ (whose dimension is $[inches]^3$).

26. A vector field v on X lifted naturally to a vector field V on T^*X preserves not only the symplectic form $d\alpha$, but the canonically defined action 1-form α. Thus, $0 = L_V\alpha = i_Vd\alpha + di_V\alpha$. This relation means that $i_V\alpha$ is the Hamiltonian function of the hamiltonian vector field V. By the definition of α, its value at $p \in T_q^*X$ on the vector $V(p)$ equals the value $p(v(q))$ of the covector p on the projection $v(q)$ of the vector $V(p)$ to the base.

30. The linearized Poisson structure is described by the commutation relations $\{Z, X\} = X$, $\{Z, Y\} = -Y$, $\{X, Y\} = C_n$, while brackets involving $C_1, \dots C_n$ are zeroes (i.e. C_k are Casimir functions). It is straightforward to check that $XY + C_nZ$ (which is the quadratic part of the Casimir function $xy + z^{n+1} + c_1z^n + \cdots + c_nz = -c_{n+1}$ of the Poisson structure before linearization) is a Casimir function of the linearized structure too. Therefore typical symplectic leaves of the linearized structure are paraboloids $XY + C_nZ = const$ in the XYZ-space when $C_n \neq 0$ (and $C_1, \dots, C_{n-1}$ take arbitrary values). When $C_n = 0$, they degenerate into $XY = 0$, the pair of intersecting planes $X = 0$ and $Y = 0$. The locus $X = Y = C_n = 0$ (where the Poisson tensor vanishes) while Z and all other C_k remain arbitrary, consists of 0-dimensional leaves. The rest of the planes (or half-planes in the real case) are the remaining 2-dimensional symplectic leaves.

Note that the union of the symplectic leaves containing the origin in their closure is the surface in the XYZ-space, given by the equation $XY + Z^{n+1} = 0$ before linearization and $XY = 0$ after it. They are not isomorphic, because former has an isolated singularity at the origin, while the latter is singular along the whole line $X = Y = 0$. This is one of the (many) features which makes the original Poisson structure not isomorphic to its linearization.

31. The Poisson structure is described by $\{z, x\} = x$, $\{z, y\} = -y$, $\{x, y\} = 3z^2 + c_2$ and vanishes at $x = y = 3z^2 + c_2 = 0$, which is the set of critical points in the family of functions $xy + z^3 + c_2z + c_3$ (in 3 variables x, y, z and with 2 parameters c_2, c_3) with zero critical values. The discriminant in the parameter space is a parametric curve $c_2 = -3z^2$, $c_3 = -z^3 - c_2z = 2z^3$. Eliminating the parameter we find $c_2^3/27 + c_3^2/4 = 0$, which is the *semicubical parabola*, the graph of both branches of the 2-valued function $c_3 = \mp\sqrt{4/27}c_2^{3/2}$.

33. A smooth action of a compact group near a fixed point is locally equivalent to its linear approximation. To prove it, take any Riemannian metric, make it action-invariant by taking the average of its transforms by the group, and use the exponential map (defined by geodesics of the metric) to equivariantly identify a neighborhood of zero in the tangent space at the fixed point with its neighborhood in the manifold. In particular, this applies to the 2-element group generated by an anti-symplectic involution. Thus, it suffices to prove the linear version of the problem. A linear involution has two complementary eigenspaces: corresponding to the eigenvalues 1 and -1. Since it changes the sign of the symplectic form, both eigenspaces must be isotropic, and hence Lagrangian. Thus, the fixed point locus of a (non-linear) anti-symplectic involution is locally diffeomorphic to a middle-dimensional subspace, and has Lagrangian tangent spaces. Thus, it is a Lagrangian submanifold.

34. The subset in M^{2n} of critical points of π is closed, hence compact. Therefore the set of the critical values of π is (compact and hence) closed in X, and the set of regular values of π is open (and, by Sard's lemma, dense in X). Thus, a connected component X_0 of this set is therefore a n-dimensional submanifold in X. It is still possible that $\pi^{-1}(X_0)$ is empty, but if not, the projection of it to X (by the restriction of π) is a proper submersion, and hence a locally trivial bundle with compact n-dimensional fibers. Since local coordinates on X_0 Poisson-commute with respect to the Poisson bracket on M, their hamiltonian vector fields (which are tangent to the the common level sets of the hamiltonians — the fibers) span isotropic n-dimensional spaces at each point, showing that the fibers are Lagrangian, and the projection $\pi^{-1}(X_0) \to X_0$ a Lagrangian fibration with compact fibers. Consequently, the fiber of the fibration is the disjoint union of a certain number $(= 0, 1, 2, \dots)$ of compact tori.

37. The linear velocity at $x \in \mathbb{R}^n$ is ωx, where $\omega \in so_n$ is an anti-symmetric matrix, and hence the kinetic energy

$$T(\omega) = \frac{1}{2}\int \rho(x)|\omega x|^2 d^n x = -\frac{1}{2}\int \rho(x)\langle \omega^t \omega x, x\rangle d^n x - \frac{1}{2}\operatorname{tr} I\omega^2,$$

where I is the symmetric "inertia" matrix, $I_{ij} = \int \rho(x) x_i x_j \, d^n x$. When $n > 3$, $\dim so_n > n$ and $\dim S^2(so_n) > \dim S^2(\mathbb{R}^n)$, implying that most left invariant Riemannian metric on SO_n cannot be obtained from an inertia matrix I. For $n = 3$, so_3 can be identified with RR^3, and as it is shown below, all left-invariant metrics on SO_3 are obtained from some inertia matrix. For any n, I can be diagonalized in a suitable orthonormal basis, i.e. $I_{ij} = 0$ for $i \neq j$, while $I_{ii} \geq 0$ can be arbitrary. Indeed, taking ρ to be distributed equally between two centrally symmetric points $\pm e_i$ on the ith coordinate axis, we can make I_{ii} an arbitrary non-negative number, keeping all other $I_{ij} = 0$, and then can take the superposition of such distributions to make I any non-negative diagonal matrix. For $I = \mathrm{diag}(d_1, \ldots, d_n)$,

$$T(\omega) = \frac{1}{2} \sum_i d_i \sum_{j \neq i} \omega_{ij}^2,$$

which for $n = 3$ yields

$$T(\omega_1, \omega_2, \omega_3) = (d_2 + d_3)\omega_1^2/2 + (d_1 + d_3)\omega_2^2/2 + (d_1 + d_2)\omega_3^2/2.$$

This is still, modulo rotations, an arbitrary[1] non-negative quadratic form in $\mathbb{R}^3$.

39. The Lagrangian can be taken in the form

$$L = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{x\dot{y} - y\dot{x}}{2}.$$

The Euler-Lagrange equations are $\ddot{x} = \dot{y}$, $\ddot{y} = -\dot{x}$, i.e. the acceleration vector is obtained from the velocity vector by clockwise rotation through 90°. Thus, each trajectory is a circle, traversed clockwise, with the speed equal to the circle's radius (and therefore with the same period 2π).

49. With $H_0 := \frac{1}{2} \int u^2 dx$ and $W = 2(\partial u + u\partial) - \partial^3$, we have $\dot{u} = W\delta H_0/\delta u = W(u) = 6uu_x - u_{xxx}$, and with $H_1 := \int \left(\frac{u_x^2}{2} + u^3 \right) dx$ and $V = \partial$ we have $\dot{u} = V\delta H_1/\delta u = V(3u^2 - u_{xx}) = 6uu_x - u_{xxx}$ as well. Next, using integration by parts, we find $\frac{d}{dt} H_0 = \int u\dot{u}dx$

$$= \int u(6uu_x - u_{xxx})dx = \int (6u^2 u_x + u_x u_{xx})dx = \int d(2u^3 + u_x^2/2) = 0,$$

i.e. H_0 is a conservation law of the KdV flow. The "higher" KdV flow defined by W and the hamiltonian H_1 is

$$\dot{u} = W\delta H_1/\delta u = (2u\partial + \partial u - \partial^3)(3u^2 - u_{xx}),$$

which after some computation can be identified with the total derivative of $u_{xxxx} - 5u_x^2 - 10uu_{xx} + 10u^3$, which is the variational derivative of $\delta H_2/\delta u$ of $H_2(u) = \frac{1}{2} \int (u_{xx}^2 + 10uu_x^2 + 5u^4)dx$.

`Remark.` The problem didn't ask to check that W is a Poisson structure, but if desired, it can be done as follows. The operator $U := u\partial - \partial u$ itself defines a *linear* Poisson structure. Indeed, if $F[u] = \int fudx$ and $G[u] = \int gudx$ are two linear functionals, then their U-Poisson bracket

$$\{F, G\} = \int (uf' + (uf)')gdx = \int u(f'g - g'f)dx,$$

[1] As was pointed out by some students, this statement is actually incorrect, since with non-negative d_i, the numbers $d_1 + d_2, d_2 + d_3, d_3 + d_1$ satisfy the triangle ineqiuality.

i.e. is again linear, with the underlying Lie algebra structure $[f,g] = f'g - g'f$. The point is that this defines a Lie algebra indeed, (and hence obeys the Jacobi identity); namely it is the Lie bracket $[g\partial, f\partial]$ of vector fields on the line. Next, the exterior 2-form on this Lie algebra given by $\omega(f,g) = \int f'''g dx = -\int f g'''$ is a 2-cocycle:

$$\omega([f,g],h) = -\int (f'g - g'f)h'''dx = \int f''gh''dx - \int g''fh''dx,$$

which after adding the cyclically permuted terms sums to zero. Thus, any linear combination of $u\partial - \partial u$ and ∂^3 defines an ("affine") Poisson structure.

50. In the basis $L_m = -z^{m+1}\partial_z = e^{imx}i\partial_x$ (where $z = e^{ix}$) of the complexified Lie algebra of polynomial vector fields on S^1, we have $[L_m, L_n] = (m-n)z^{m+n+1}\partial_z = (m-n)L_{m+n}$ as required. Clearly, L_{-1}, L_0, L_1 span a Lie subalgebra, which is in fact isomorphic to $sl_2(\mathbb{C})$, the Lie algebra of the group $PGL_2(\mathbb{C})$ of automorphisms $z \mapsto (az+b)/(cz+d)$ of the Riemann sphere $\mathbb{C}P^1$. The group does contain the subgroup $SL_2(\mathbb{R})$ of projective transformations of the real projective line $\mathbb{R}P^1 \equiv S^1$. But this subgroup preserves the real line (and the upper half-plane) on the complex z-plane, not the circle $|z| = 1$. The subgroup preserving the circle (and the unit disk enclosed by it) is rather identified with another real form of SL_2: the (quotient by $\pm I$) group $SU(1,1)$ of automorphisms of $\mathbb{C}^2$ preserving an Hermitian form of signature $(+,-)$.

The 2-form ω on $Vect(S^1)^{\mathbb{C}}$ is a 2-cocycle because

$$\omega([L_m, L_n], L_k) + \text{cycle} = (m-n)(m+n)^3\delta_{m+n+k,0} + \text{cycle} = 0$$

by elementary algebra. On $Span(L_{-1}, L_0, L_1)$, it is the coboundary of the linear form μ taking values $0, 1/2, 0$ on the basis (as it is not hard to check). However, there is a general argument (based on semisimplicity of sl_2) showing that the cocycle must be a coboundary. More explicitly, the fact that $SL_2(\mathbb{C})$ has another, compact real form SU_2 allows one (via integration over SU_2) to construct an Ad-invariant positive-definite inner product on the central extension defined by the cocycle, and then split the central extension into the direct sum of Lie subalgebras by taking the orthogonal complement to the center.

51. The given 2-cochain $\omega(v(x_0), w(x_0))$ is the evaluation of the Poisson bracket $\{h_v, h_w\}$ of hamiltonians of the hamiltonian vector fields v and w at the point x_0. With the choice of the hamiltonians to vanish at x_0 (unique since M is connected) the cochain describes (or comes from) the central extension of the Lie algebra of hamiltonian vector fields by the Poisson algebra of Hamilton functions:

$C(v,w) = \{h_v, h_w\}(x_0) - h_{[v,w]}(x_0)$. It is a 2-cocycle due to the Jacobi identity in the Poisson algebra. On a compact M^{2n}, the cocycle is a coboundary, because the central extension splits by another normalization of the hamiltonians: $\int_M h\omega^n = 0$. Namely,

$$\int_M \{h_v, h_w\}\omega^n = \int_M (L_v h_w)\omega^n = \int_M L_v(h_w\omega^n) = \int_M di_v(h_w\omega^n) = 0,$$

i.e. $\{h_v, h_w\} = h_{[v,w]}$ since both sides have zero average.

52. If $t \to \epsilon(t) \in G$ is a curve, $\epsilon(0) = e$, representing a tangent vector $a \in T_eG = \mathfrak{g}$, then the action of this curve on G by *left* translations: $(t,g) \mapsto \epsilon(t)g$ yields a family of curves invariant with respect to *right* translations. That is, the action of G on itself by left translations corresponds to the embedding $\mathfrak{g} \in Vect(G)$ by right-invariant vector fields, v_a. Respectively, the moment map $T^*G \to \mathfrak{g}^*$ (defined by $T_g^*G \ni p \mapsto [a \mapsto p(v_a(g))]$ is given by the projection $\mathfrak{g}^* \times G \to \mathfrak{g}^*$ in the trivialization $T^*G = \mathfrak{g}^* \times G$ of the cotangent bundle by right translations.

53. For a connection $\nabla = d + A\wedge$ where $A \in \Omega^1(\Sigma;\mathfrak{g})$, its curvature $\nabla^2 = dA + A \wedge A$ which in local coordinates x_1, x_2 on Σ and for $A = A_1dx_1 + A_2dx_2$ equals $[\partial_{x_1}A_2 - \partial_{x_2}A_1 + A_1A_2 - A_2A_1]dx_1 \wedge dx_2$. Here $A_1A_2 - A_2A_1$ is the commutator of $A_i \in \mathfrak{g}$ in the adjoint (or any other) representation, or equivalently, the action of $[A_1, A_2] \in \mathfrak{g}$ in that representation. Given $a \in \Omega^0(\Sigma;\mathfrak{g})$, consider the hamiltonian $H_a : \nabla \mapsto \int_\Sigma \mathrm{tr}(a \otimes \nabla^2)$ (of degree ≤ 2 on the affine space of connections). The value of the differential $d_\nabla H_a$ on a tangent vector $B \in \Omega^1(\Sigma;\mathfrak{g})$ equals $\int_\Sigma \mathrm{tr}(a \otimes \nabla B)$ (where in local coordinates $\nabla B = (\partial_{x_1}B_2 - \partial_{x_2}B_1 + [A_1, B_2] - [A_2, B_1])\, dx_1 \wedge dx_2$.) We have: $d\,\mathrm{tr}(a \otimes B) = \mathrm{tr}(\nabla a) \otimes B + \mathrm{tr}\, a \otimes \nabla B$, which is the Leibniz rule for the connection operator (it acts on sections of all vector bundles associated with the adjoint one) taking into account that $\nabla = d$ on scalar-valued functions (where the values of tr belong). Since $\partial\Sigma = \emptyset$, we can rewrite $(d_\nabla H_a)(B)$ as $-\int_\Sigma \mathrm{tr}(\nabla a) \otimes B$, i.e. (taking into account Arnold's sign convention) the hamiltonain vector field generated by the Hamilton function H_a is $\nabla \mapsto \nabla a$. This is exactly the infinitesimal action of the Lie algebra of currents $\Omega^0(\Sigma;\mathfrak{g})$ corresponding to the action of the gauge group on connections: $\nabla = d + A \mapsto g^{-1}\nabla g = d + g^{-1}dg + g^{-1}Ag$. Thus, the gauge action is Poisson, with the moment map $\nabla \mapsto \nabla^2$.

55. The action of the torus of diagonal unitary matrices in $\mathbb{C}^{n+1}$ (considered as a real symplectic space of dimension $2n+2$) is generated by the hamiltonians $|z_i|^2$. The action by the unitary scalars is given by their sum: $|z_0|^2 + \cdots + |z_n|^2$ (which is U_{n+1} invariant). The symplectic reduction at the unit (or any other positive) level of the latter hamiltonian yields the symplectic quotient $\mathbb{C}P^n = S^{2n+1}/U_1$, equipped with a U_n-invariant symplectic form. It is unique up to a scalar factor (depending on the level of the hamiltonian and on the normalization of the initial symplectic form in $\mathcal{C}^{n+1}$) and, when appropriately normalized, coincides with the Fubini-Study one. The action of the torus descends to the symplectic quotient and is given by the hamiltonians $(z_0 : \cdots : z_n) \mapsto |z_i|^2|_{|z|^2=const}$, $i = 0, \ldots, n$. So, the image of the moment map is the simplex $x_0 + \cdots + x_n = const > 0$, $x_i \geq 0$. Its vertices are the images of the fixed points of the torus action which are the coordinate lines in $\mathbb{C}^{n+1}$ (which are the common eigenvectors of all the diagonal matrices).

56. Here is Atiyah's argument:

By considering the representations of T on the normal bundles of the components Z_α of the common critical set Z we get a finite set of characters, and their kernels give a finite set of codimension-one subtori of T. Taking intersections these generate a finite lattice [poset – A.G.] of sub-tori. Without essential loss of generality, we may assume T acts effectively on M. Then the minimal non-zero elements of our lattice will be circles $S_1, \ldots, S_k$ and the quotient $(n-1)$-torus T/S_i acts effectively on the components Σ_{ij} of the fixed-point set of S_i. Restricting the moment map f to Σ_{ij} we see therefore that its image in $\mathbb{R}^k$ lies in the hyperplane $\sum \lambda_{ri} x_r = constant$, where $\phi = \sum \lambda_{ri} f_i$ is the Hamiltonian corresponding to S_i. Moreover $f(\Sigma_{ij})$ will contain (and is spanned by) the subset of $\{c_\alpha\}$ corresponding to the components of Z lying in Z_α. Thus the union of all these hyperplanes contains the set of critical values of the map f. A bounding face of the convex polytope $f(M)$ must arise from a maximal or minimal component of the corresponding function ϕ.

57. The equality follows from the Duistermaat–Heckman formula. According to it, the integral $\int_M e^{\sum u_i H_i} \frac{\omega^{\wedge n}}{n!}$ is equal to the principal part of its stationary phase asymptotics. The Hamiltonians $H_j = |z_j|^2 = (p_j^2 + q_j^2)$ (where $z_j = p_i + \sqrt{-1} q_i$) generate commuting $\mathbb{Z}$-periodic hamiltonian flows $(z_j \mapsto e^{2\pi\sqrt{-1}t} z_j)$ with respect to the symplectic form

$$\omega = \frac{1}{2\pi\sqrt{-1}} \sum_{i=0}^{n} d\bar{z}_i \wedge dz_i = \frac{1}{\pi} \sum_{i=0}^{n} dp_i \wedge dq_i.$$

The fixed points of the T^{n+1}-action on $proj(\mathbb{C}^{n+1})$ are the coordinate axes $span(e_i) \in \mathbb{C}^{n+1}$. In the affine chart $z_0 = 1$ near the critical point

$z_1 = \cdots = z_n = 0$ we have

$$\sum u_j H_j = \frac{u_0 + \sum_{j>0} u_j |z_j|^2}{1 + \sum_{k\neq 0} |z_k|^2} = u_0 - \sum_{j\neq 0} (u_0 - u_j)|z_j|^2 + o(|z|^2).$$

By evaluating a $2n$-dimensional Gaussian integral, we find the principal term of the stationary phase asymptotics:

$$\int_{\mathbb{C}^n} e^{\sum u_j H_j} \frac{1}{\pi^n} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n \sim \frac{e^{u_0}}{\prod_{j=1}^n (u_0 - u_j)}.$$

Such principal terms for all fixed points form the residue sum:

$$\sum_{i=0}^n \frac{e^{u_i}}{\prod_{j\neq i}(u_i - u_j)} = \frac{1}{2\pi\sqrt{-1}} \oint \frac{e^p dp}{\prod_{j=0}^n (p - u_j)}.$$

Note however, that the equality is a result of a lucky (or clever) choice of normalization of the symplectic form. Namely, the LHS of the Duistermaat–Heckman formula in the limit $u = 0$ yields the symplectic volume $\int_M \omega^{\wedge n}/n!$ of the manifold and scales as c^n under $\omega \mapsto c\omega$. (Given the torus action, the hamiltonians scale the same way: $H_i \mapsto cH_i$, while their Hessians at the critical points — whose square roots appear in the denominators of the RHS — remain unchanged.) When the symplectic form on $\mathbb{C}P^n$ is obtained as the symplectic reduction from $\mathbb{C}^{n+1}$ at a level $|z|^2 = c$, the (cohomology class of the) symplectic form on $\mathbb{C}P^n$ depends linearly on c. To find out what cohomology class corresponds to our choice of normalization, note that in the limit $u = 0$ our residue integral turns into

$$\frac{1}{2\pi\sqrt{-1}} \oint \frac{e^p dp}{p^{n+1}} = \frac{1}{n!}.$$

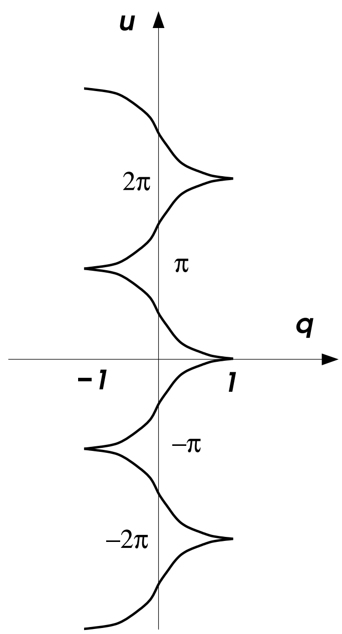
Thus, $\int_{\mathbb{C}P^n} \omega^{\wedge n} = 1$, implying that the cohomology class of our symplectic form is a generator of the integer cohomology $H^2(\mathbb{C}P^n; \mathbb{Z})$ (more precisely, the one which integrates to 1 over a projective line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ equipped with the complex orientation).

58. If α is a contact form on a 3-fold defining locally a given contact structure, then $\alpha \wedge d\alpha$ is a volume form, and defines orientation locally. When α is multiplied by a non-zero function f, the volume form $\alpha \wedge d\alpha$ is multiplied by $f^2 > 0$, and hence defines the same localorientation. Thus, the orientation does not depend of the local choice of α, but only on the contact structure itself. The same works on contact manifolds of dimension 3 modulo 4.

61. In the parameterization $p = \sin t$, $q = \cos t$, we have

$$u = \int_0^t \sin\xi \, d\cos\xi = -\int_0^t \sin^2\xi \, d\xi = \frac{\sin 2t}{4} - \frac{t}{2} = \frac{1}{2}\left(\pm q\sqrt{1-q^2} - \arccos q\right).$$

The front is the graph of a multivalued function whose derivative is $\pm\sqrt{1-q^2}$. Its graph is shown on the figure.



64. Since differentiations ∂_{q_i} (multiplications by q_j) commute with differentiations (resp. multiplications), the property $[D_m, D_n] \in \mathcal{D}_{m+n-1}$ follows from $[\partial_{q_i}, f(q)] = \partial f/\partial q_i \in \mathcal{D}_0$ by induction on the order of differential operators. Modulo $\mathcal{D}_{m+n-1}$, we have: $D_m(\partial_q, q)D_n(\partial_q, q) \equiv D_m(p,q)D_n(p,q)|_{p=\partial_q}$, i.e. the product in $Gr(\mathcal{D})$ coincides with the multiplication of functions on T^*X polynomial in p. Furthermore, modulo $\mathcal{D}_{m+n-2}$,

$$[D_m(\partial_q, q), D_n(\partial_q, q)] \equiv$$
$$\sum_{i,j}\left[\frac{\partial D_m(p,q)}{\partial p_i}\frac{\partial D_n(p,q)}{\partial q_j} - \frac{\partial D_n(p,q)}{\partial p_i}\frac{\partial D_m(p,q)}{\partial q_j}\right]_{p=\partial_q}[\partial_{q_i}, q_j].$$

This coincides with the Poisson bracket $\{D_m(p,q), D_n(p,q)\}$ due to the Heisenberg commutation relations $[\partial_{q_i}, q_j] = \delta_{ij}$.

66. The characteristics of Burgers' equation $u_\tau = uu_x$ (which is expresses by the hypersurface $p_0 - up_1 = 0$ in 1-jet space with the contact structure $du = p_0 d\tau + p_1 dx$ satisfy the ODE system

$$\dot{p}_0 = p_0 p_1,\ \dot{p}_1 = p_1^2,\ \dot{\tau}_0 = 1,\ \dot{x} = u,\ \dot{u} = p_0 - up_1,$$

The equations imply $\tau(t) = t$ (assuming that $\tau(0) = 0$), $\dot{u} = 0$ (i.e. u is constant along the characteristics), and $x(t) = x(0) + ut$, meaning that the characteristics project to the (u, τ, x)-space to the lines $u = u(0), x = x(0) + u(0)\tau$. The initial conditions (at $\tau = 0$ and any x) read $u(0, x) = v(x)$, $p_1(0, x) = v'(x)$, $p_0(0, x) = v(x)v'(x)$. Therefore $u(\tau, x) = v(x(0))$ where $x(0)$ as a function of x and τ is determined by the implicit equation $x = x(0) + v(x(0))\tau$. In particular, it is possible that through the same point (τ, x), several "rays" $x = x(0) + v(x(0))\tau$ are passing each bringing its own value $v(x(0))$ of the (then multiple-valued) function u. (This equation can serve as a model of a collisionless non-interacting medium filled with particles, each moving with its own velocity (which is distributed according to the density $u(\tau, x)dx$ at the moment τ), and bringing its velocity to whatever position the particle occupies at a subsequent moment.)