Math 215B. Spring 2020. Homework answers

Homework 7.

1. This exercise is the cohomological version of Proposition on p. 346 of Fomenko-Fuchs. So, the solution is contained in the diagram on bottom of p. 346, but all arrows need to be reversed.

2. By WHE, we may assume that \( X, Y \) are cell spaces. Then

\[
\tilde{H}^*(X \vee Y; G) = H^*(X \sqcup Y, x_0 \sqcup y_0; G) = \tilde{H}^*(X; G) \oplus \tilde{H}(Y; G).
\]

The restriction homomorphism \( \tilde{H}^*(X \times Y; G) \to \tilde{H}^*(X \vee Y; G) \) is surjective, because the composition

\[
H^*(X, x_0; G) \to H^*(X \times Y, x_0 \times y_0; G) \to H^* \to \tilde{H}^*(X \vee Y; G) \to H^*(X, x_0; G),
\]

where the first map is induced by the projection \( X \times Y \to X \), and the last one by the inclusion \( X \subset X \vee Y \), is the identity, and the same is true for \( Y \) instead of \( X \). From the exact sequence of the pair \((X \times Y, X \vee Y)\) it follows now, that \( \tilde{H}^*(X \# Y; G) \) is the kernel of the above restriction homomorphism. When \( G \) is a field, the Kunneth formula \( H^*(X \times Y; G) = H^*(X; G) \otimes H^*(Y; G) \) implies that the kernel \( \tilde{H}(X \# Y; G) = \tilde{H}(X; G) \otimes \tilde{H}(Y; G) \).

3. If \( a_i \in H^n(X; \Pi), i = 1, 2 \), are induced from the fundamental class \( e_n \in H^n(K(\Pi, n); \Pi) \) by characteristic maps \( f_i : X \to K(\Pi, n) \), then, replacing \( K(\Pi, n) \) with \( \Omega K(\Pi, n + 1) \), we obtain \( \Sigma a_i = (\Sigma f_i)^*(e_{n+1}) \). The operation of composition of loops gives rise, equivalently, to

\[
\Sigma X \xrightarrow{\pi} \Sigma X \vee \Sigma X \xrightarrow{\Sigma f_1 \vee \Sigma f_2} K(\Pi, n + 1),
\]

inducing \( e_{n+1} \mapsto \Sigma a_1 + \Sigma a_2 \mapsto \Sigma (a_1 + a_2) \). Here \( \pi : \Sigma X \to \Sigma X \vee \Sigma X \) is the usual “pinching” of the middle section of the suspension. In this sense, the composition of loops (together with the suspension isomorphism) induces the addition \((a_1, a_2) \mapsto a_1 + a_2 \) in \( H^n(X; \cdot) \) in a way similar to the addition in homotopy groups: by wedging together two maps into one.

Now, if \( \phi \) is any cohomological operation \( H^{n+1}(\cdot; \Pi) \to H^{n+r+1}(\cdot; G) \), then (due to the naturality with respect to \( \pi \))\(^1\)

\[
\phi(\Sigma a_1) + \phi(\Sigma a_2) = \pi^* \left[ \phi(\Sigma a_1) \oplus \phi(\Sigma a_2) \right] = \phi \pi^* [\Sigma a_1 \oplus \Sigma a_2] = \phi(\Sigma(a_1 + a_2)).
\]

If in addition \( \phi \) is stable, then \( \phi(a_1) + \phi(a_2) = \phi(a_1 + a_2) \).

\(^1\)For example, \( \phi(a) := a^p \) generally speaking is not additive, but \((\Sigma a_1 + \Sigma a_2)^p = 0 = (\Sigma a_1)^p + (\Sigma a_2)^p \) since the cup-product in \( \tilde{H}^*(\Sigma X; \Pi) \) is trivial.
4. Let \( \binom{A}{n} := A(A-1) \cdots (A-n+1)/n! \) denote the Taylor coefficient at \( x^n \) of \( (1 + x)^A \). Then \( \binom{m-k+j-1}{j} = \pm \binom{k-m}{j} \), and so mod 2 the required equality can be written more elegantly as

\[
S^k w_m = \sum_{j=0}^k \binom{k-m}{j} w_{k-j} w_{m+j}, \quad \text{where } k \leq m.
\]

Put \( S^a = \sum_{k \geq 0} u^k S^k \), \( w(t) = \sum_{m \geq 0} t^m w_m = \prod_j (1 + tx_j) \). Then by Cartan’s theorem

\[
S^a w(t) = \prod_j (1 + tx_j + ut x_j^2) = \prod_j (1 + \xi x_j) \prod_j (1 + \eta x_j) = w(\xi) w(\eta),
\]

where \( \xi + \eta = t \), \( \xi \eta = ut \) are Vieta’s formulas. Using them, rewrite the weight \( u^k t^m \) of \( S^k w_m \) in \( S^a w(t) \) as

\[
(\xi \eta)^k (\xi + \eta)^{m-k} = \sum_{i=0}^{\left\lfloor \frac{m-k}{2} \right\rfloor} \binom{m-k}{i} (\xi \eta)^{k+i} \frac{\xi^{m-k-2i} + \eta^{m-k-2i}}{\epsilon(m-k-2i)},
\]

where \( \epsilon(0) := 2 \) and \( \epsilon(l) := 1 \) for \( l > 0 \). For \( a \leq b \) the coefficient at \( (\xi \eta)^{a} (\xi^{b-a} + \eta^{b-a})/\epsilon(b-a) \) in \( w(\xi) w(\eta) \) is then found as

\[
w_a w_b = \sum_{i=0}^{a} \binom{b-a+2i}{i} S^a i w_{b+i}.
\]

(Here \( k + i = a \), \( m - k - 2i = b - a \), and hence \( k = a - i \), \( m = b + i \).)

The relations, interpreted in the matrix form, are inverse to those we are purporting to prove. Showing that the corresponding coefficient matrices are inverse amounts to verifying that mod 2

\[
\sum_{i+j=l} \binom{k-m}{j} \binom{m-k+2l}{i} = 1 \text{ for } l = 0 \text{ and } 0 \text{ for } l > 0.
\]

The binomial sum is the Taylor coefficient at \( x^l \) of \( (1 + x)^{k-m} (1 + x)^{m-k+2l} = (1 + x)^{2l} \), i.e. \( \binom{2l}{l} \). From \( (1 + x)^{2l} = (1 + x)^l (1 + x)^l \) and \( \binom{l}{s} = \binom{l}{l-s} \), we find:

\[
\binom{2l}{l} = \sum_{s=0}^{l} \binom{l}{s}^2 = \sum_{s=0}^{l} \binom{l}{s}^2 = 2^l \mod 2.
\]